A simple method of measuring the volumetric flow of a stream

by Nick Doe

One of the several methods of calculating the volumetric flow of a stream is to measure the transverse cross-sectional area of the water and multiply this by its velocity. This is however, easier said than done. The transverse profile of a stream varies from point-to-point along its course; it varies seasonally with the water level; and the velocity of the water varies both with distance from the bank and the depth of the water below the surface.

While professional surveyors have the capability of measuring these parameters in great detail, this is, even for them, a time-consuming exercise. The official USGS method for example requires, or used to require, I may be out-of-date, at least twenty measurement points across the stream.

Since the flow of the stream is a product of cross-sectional area and velocity, there is no point in measuring just one of these parameters with painstaking accuracy and using a rough estimation of the other. I have seen examples of an explanation of the method where measuring the cross-sectional stream depth is measured in detail and this is combined with a measurement of velocity as measured by the time a floating object travels a measured reach. Not only does this assume that the cross-sectional area is constant over the reach—or at least has an average cross-sectional area close to the measured one— it also assumes that the float follows a steady course down the stream at a rate that bears a reasonably constant relationship with the average velocity of the water. In my experience with the small streams on Gabriola and across on Vancouver Island in the Nanaimo River watershed, it seldom does.

In this note I want to tackle the problem of measuring volumetric flow quickly, yet reasonably accurately, without a multiplicity of depth measurements and runs with a float. The emphasis here is on keeping the number of field measurements to a minimum and not worrying about any mathematical complexity engendered by the fact that the data is sparse. Spread-sheets can easily and painlessly deal with any complicated math required to compensate for lack of detailed measurements.

Form ratios (FR)

All of the water courses on Gabriola are creeks; we have nothing that you could call a river. This means that their *form ratios*, that is their width-to-depth ratios, ($W:D_0$), tend to be lower than the average for water courses. In excavated drainage ditches, the form ratio can be as low as 3:1 (34º transverse slopes) when full, whereas in creeks and partially full ditches it is more commonly around 10:1 (11º transverse slopes), and in substantial rivers it can be more than 100:1 (1º transverse slopes).[1]

[1] Ditches often give a false impression that their form ratios are low because their banks are high; however, “bank” throughout this article means the waterline—the point of contact of the water and the dry land—which, except in flood conditions, is much closer to the bottom of the ditch than the top. False Narrows has a form ratio fixed by ice-scoured bedrock of about 67:1 (1.7º).

Streams that flow over alluvium rich in coarse sand generally have a high form ratio; they are broad and shallow[2] whereas streams that flow over clay-rich soil have a lower form ratio because they do not erode their banks so readily. On Gabriola, many creeks flow in narrow channels eroded into clay-rich, heavily-compacted, lodgement till, and so when the creeks are full, they have low form ratios. Other creeks however have a shallow saucer-like profile and high form ratios because they flow in shallow depressions in the sandstone bedrock.

Area estimates

Essentially the problem of measuring the cross-sectional area is one of characterising the profile with as few sampling points as possible.

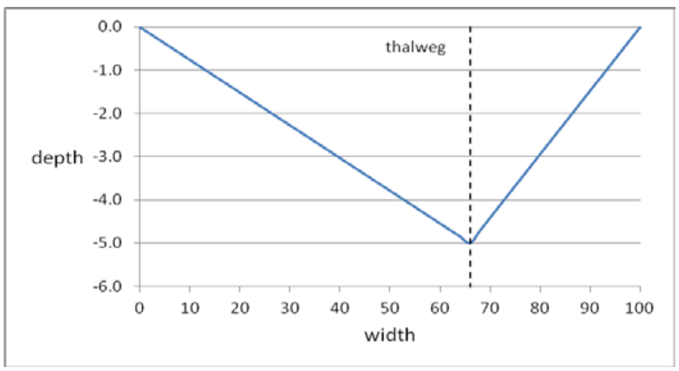


Figure 1: The triangular characterization of the stream profile based on width W, thalweg depth $-D_0$, and thalweg distances from the banks T_0 and $W-T_0$.

Thalweg stream depth

Of the two measurements that it is essential to make, one is the depth of the deepest point in the stream, which is known as the thalweg ("way to the valley") depth, D_0. In all but the very simplest methods of assessing the cross-sectional area, it is also necessary to know the distance of the thalweg T_0 from one of the banks, say the left bank.[3]

The mathematical procedures for computing the cross-sectional area to the left and right of the thalweg are the same—you just have to mirror image the profile of the right side making the bank the starting point, and so in this analysis, I'll only deal with the area to the left of the thalweg and leave the details of adding in the right side until the finish.

The simplest method

The simplest mathematical representation of the profile of the bottom of the stream is then:

$$y(x) = \alpha + \beta x \qquad 0 \leq x \leq T_0 \tag{1}$$

where $y(x)$ is the stream depth measured negatively,[4] at distance x from the left bank, and T_0 is the distance of the thalweg from the left bank.

We must have $y(0) = 0$ and $y(T_0) = -D_0$ and using these boundary conditions in (1) gives us:

$$y(x) = -D_0\, x/T_0 \qquad 0 \leq x \leq T_0 \tag{2}$$

The cross-sectional area A_L is then:

$$A_L = \int -y(x)\, dx$$

$$= [D_0 x^2/2T_0]$$

$$= 0.50\, D_0 T_0 \tag{3}$$

integration from x=0 to T_0.

The slope of the stream bed is:

[2] Sandy beaches have a slope of about 6º implying a form factor of around 2/tan(6º) = 20:1.

[3] Left is usually taken to mean relative to an observer looking downstream, but in this article it doesn't matter which bank you use. I just think of it as the bank depicted on the left side of the page.

[4] The word "depth" can be confusing; I try to say "stream depth" when I mean the distance between the bed and the surface and just "depth" when I mean a selected distance below the surface.

$dy/dx = –D_0/T_0$

For this particularly simple method, it is not actually necessary to measure T_0 because the cross-sectional area of the right side of the stream is:

$A_R = 0.50\ D_0(W–T_0)$

where W is the width of the stream at the surface, and combining the two cross-sections gives:

$A = 0.50\ D_0W$

Does it work?

Strong support for the adequacy of this method is provided by a paper by Robison and Beschta (1989).[5] They measured 334 cross-sections and compared the area with that predicted by the double "triangle method",[6] which is Eq.4.

The profile of the stream in this method can be, but doesn't have to be, construed to be V-shaped with the point of the V being at the thalweg. Robison and Beschta 1989 found, on average, close correlation between the measured and predicted values of cross-sectional area.

Very few streams actually have a V-shaped profile, but this is not a weakness of the method as it applies to any stream with a profile in the form:

$y(x) = –D_0x/T_0 + \varepsilon(x) \quad 0 \le x \le T_0$

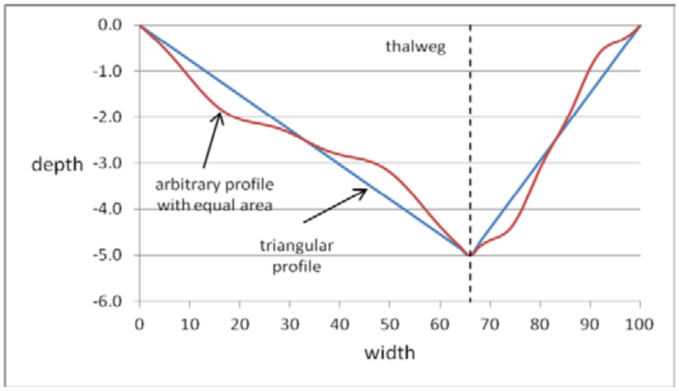


Figure 2: Actual stream profiles need not be triangular for the triangle method to give the right cross-sectional area, but over- and under-estimates of depth have to balance.

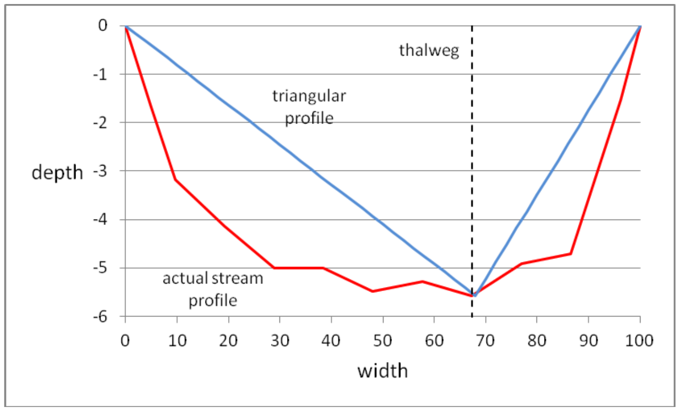


Figure 3: An example of a real stream profile whose cross-sectional area is not very well estimated by the triangle method. The depth and widths have been scaled; the stream (Lamprey River at New Market NH USA) was 32 m wide and 1.77 m deep. The estimated cross-sectional area error is –35%.

where $\varepsilon(x)$ is any function that complies with the conditions:
$\varepsilon(0) = 0$, $\varepsilon(T_0) = 0$, and
$\int \varepsilon(x)\,dx = 0$ integration from x=0 to T_0.

The last requirement is simply that under- and over-estimations of the stream depth compared with the V-shaped estimations be equal.

An example function yielding an infinite variety of possibilities is:

[5] E. George Robison and Robert L. Beschta, *Estimating stream cross-sectional area from wetted width and thalweg depth*, Physical Geography, 10, 2, pp.190–198, 1989.

[6] So-called because the area is the area of the triangles bounded by the surface, a vertical line through the thalweg, and straight lines joining the wetted banks with the thalweg on the stream bottom.

$\varepsilon(x) = \Sigma a_n \sin(2\pi\, nx/T_0)$
where n is any integer, a_n are arbitrary constants, and the sumation is for all $a_n \neq 0$.

Robison and Beschta's close correlation between measured cross-sectional area and predicted values by the triangle method, Eq.3, comes with one crucial proviso:

> "…while t-tests and regression analysis showed a close correspondence between the two methods, the triangle method may yield a cross-sectional area that is considerably in error at a single cross-section…"

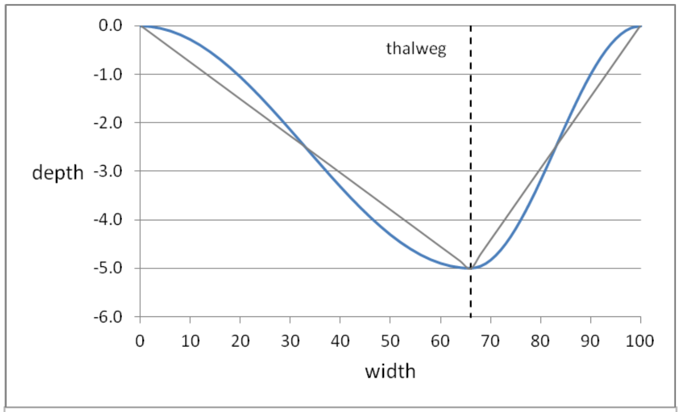


Figure 4: Using a simple trignometric function (Eq.8) to model the stream profile. The cross-sectional area is exactly the same as in the triangular method but the profile is more realistically flat at the thalweg.

Their good result was only achieved by taking several cross-sectional measurements and averaging, which is something we don't want to have to do.

Illustration of the weakness of the method for single sampling points was provided by an analysis of a real stream profile, Fig. 3, which shows an error of –35%.

A simple improvement?

A weakness of Eq.1 is that it does not require the slope of the stream bed to be zero at the thalweg, which it must be by definition. One way of overcoming this objection is to modify Eq.1 to:

$$y(x) = \alpha + \beta x + \gamma x^2 \quad 0 \leq x \leq T_0 \qquad (5)$$

with the conditions

$y(0) = 0$, $y(T_0) = -D_0$, and $dy/dx\ (T_0) = 0$

Using these gives us:

$$y(x) = -2D_0x/T_0 + D_0x^2/T_0^2 \qquad 0 \leq x \leq T_0 \qquad (6)$$

The cross-sectional area A_L is then:

$$A_L = \int -y(x)\, dx = [D_0x^2/T_0 - D_0x^3/3T_0^2] = 0.67\ D_0T_0$$

integration from x=0 to T_0.

The slope of the stream bed is:

$$dy/dx\ (x) = (-2D_0/T_0)(1 - x/T_0)$$

and its average slope, $-D_0/T_0$, is the same as for the triangle method. Fig.4.

The change in cross-sectional area as a result of requiring the stream bed to be flat at the thalweg is a change in the constant relating area to the product D_0T_0 from 0.50 to 0.67, which may be fine for some streams, but is seriously at odds with the Robison and Beschta observations. We need something that is more convincing to improve the accuracy.

A more better improvement

One approach to improving Eq.5 is to make it a higher-order polynomial and we can effectively do this by using trigonometric functions, in the same way as is used in Fourier analysis. Trigonometric functions are polynomials of infinite order, and have the gentle curvatures that seem a good fit for a stream profile.

A good starting point is:

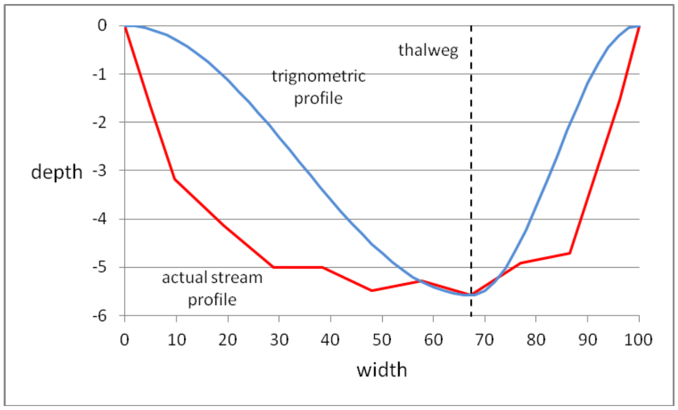


Figure 5: An example of a real stream profile whose cross-sectional area is not very well estimated by the trignometric method either, see Fig. 3. The estimated cross-sectional area error is still –35%.

$$y(x) = \alpha\{\cos(\pi x/T_0) - 1\} \qquad 0 \le x \le T_0 \qquad (7)$$

The minimal conditions:

$y(0) = 0$
$y(T_0) = -D_0$
$dy/dx\ (T_0) = 0$

require $\alpha = D_0/2$. Hence

$$y(x) = D_0/2\ \{\cos(\pi x/T_0)-1\} \qquad 0 \le x \le T_0 \qquad (8)$$

$$dy/dx\ (x) = -D_0\pi/2T_0 \sin(\pi x/T_0)$$

$$A_L = \int -y(x)\, dx$$

$$= [-D_0T_0/2\pi \sin(\pi x/T) + D_0x/2]$$

$$= 0.50\ D_0T_0 \qquad (9)$$

integration from $x=0$ to T_0.

We can now look to see why in some cases Eq.8 is not a good description of the stream profile, Fig. 5.

Clearly, although the change from a V-shaped profile to a more U-shaped profile is an improvement, there is no improvement in the accuracy of the estimate of cross-sectional area.

Scaling

The scaling can be improved in a simple fashion by following Fourier-like procedures and modifying Eq.7 thus:

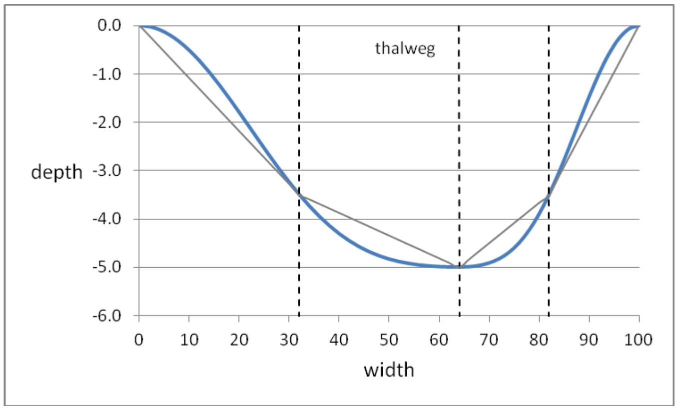
$$y(x) = \alpha\{\cos(\pi x/T_0) - 1\} + \beta\{\cos(2\pi x/T_0) - 1\} \qquad 0 \le x \le T_0 \qquad (10)$$

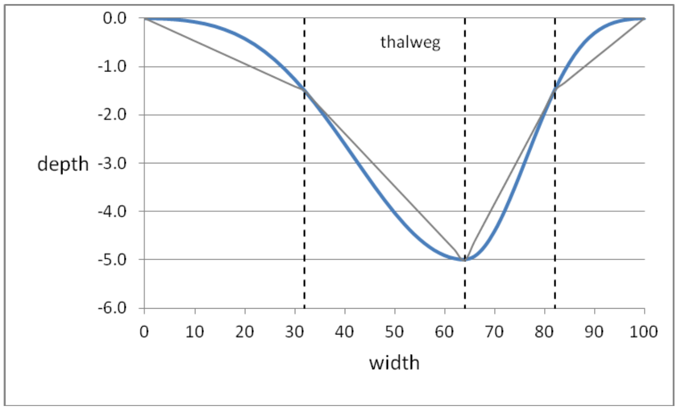
$$dy/dx\ (x) = -\alpha\ \pi/T_0 \sin(\pi x/T_0) - 2\pi\beta\ /T_0 \sin(2\pi x/T_0)$$

We must have:

$y(0) = 0$; we do
$y(T_0) = -D_0$; hence $\alpha = D_0/2$
$dy/dx\ (T_0) = 0$; we do

hence:





Figures 6 & 7: Using a simple trignometric function (Eq.11) to model the stream profile. By making an additional measurement each side, a variety of profiles can be accomodated. In Fig.6 the profile is U-shaped. In Fig7, it is more V-shaped

$$y(x) = D_0/2\{\cos(\pi x/T_0) - 1\} + \beta\{\cos(2\pi x/T_0) - 1\} \quad (11)$$

and the cross-sectional area becomes:

$$\int -y(x)\,dx = [-D_0T_0/2\pi \sin(\pi x/T_0) + D_0x/2 - \beta T_0/2\pi \sin(2\pi x/T_0) + \beta x]$$

$$= 0.50\, D_0T_0 + \beta T_0 \quad (12)$$

integration from x=0 to T_0.

This allows for the same average as measured by Robison and Beschta, but also allows for departures from this average by letting $\beta \neq 0$.

The correction

The value of β? This can only be determined by making another stream depth measurement between x = 0 and x = T_0, say at x = T_1 where the depth, $y(x_1)$, is $-D_1$

We then have:

$$\beta = \beta U / \beta L$$
$$\beta U = -D_0/2\ \{\cos(\pi T_1/T_0) - 1\} - D_1$$
$$\beta L = \cos(2\pi T_1/T_0) - 1$$

If $T_1 = T_0/2$, that is the additional sample stream depth is measured halfway between the bank and the thaweg—it doesn't have to be there, but that's a good choice—then:

$$2\beta = D_1 - D_0/2$$

If $D_1 = D_0/2$, implying a V-shaped profile, then no changes are necessary to Eq.3 or Eq.9. If $D_1 > D_0/2$, implying a more incised (rectangular) profile, then you have to increase the cross-sectional area, Eq.11.

If $D_1 < D_0/2$, implying a profile that extends only shallowly from the banks, then you have to decrease the cross-sectional area, Eq.11.

One more tweak

Looking at Fig. 5, we can see that a likely departure from the theoretical curve from the profile of real streams is that the slope of the theoretical curve at the bank is always zero. Eq.10 implies $dy/dx(0) = 0$. Although this may be accurate for some streams that have overflown their banks, it is not true in general. Many creeks have steep drop-offs as you move into them from the shore.

We can improve this, but it should noted that from the point of view of obtaining an accurate estimate of the cross-sectional area, fiddling around with the profile near the banks is not going to change the estimate of area by very much. The contribution to the cross-sectional area of water near the banks of the stream is going to be small, so errors in the profile there are of less importance than errors in the profile nearer the thalweg. Nevertheless having an unchangeable value for $dy/dx(0)$ of zero is probably not a good idea.

We can correct this by re-writing Eq. 10 as:

$$y(x) = \alpha\{\cos(\pi x/T_0) - 1\} + \beta\{\cos(2\pi x/T_0) - 1\} + \gamma\{\sin(\pi x/2T_0)\} \quad (11)$$

$$dy/dx\,(x) = \alpha\pi/T_0\{-\sin(\pi x/T_0)\} + 2\pi\beta/T_0\{-\sin(2\pi x/T_0)\} + \pi\gamma/2T_0\{\cos(\pi x/2T_0)\}$$

We must have:

$y(0) = 0$; we do
$y(T_0) = -D_0$; hence $\alpha = D_0/2 + \gamma/2$
$dy/dx\,(T_0) = 0$; we do

but instead of $dy/dx\,(0) = 0$; we have:

$$dy/dx\,(0) = \pi\gamma/2T_0$$

If we assign a value to $dy/dx\,(0)$, say $-\kappa D_0/T_0$, where κ is a dimensionless constant of our own choosing, then:

$$\gamma = -(2\kappa/\pi)\, D_0$$
$$\alpha = D_0(1/2 - \kappa/\pi)$$

$$y(x) = D_0(1/2 - \kappa/\pi)\{\cos(\pi x/T_0) - 1\} + \beta\{\cos(2\pi x/T_0) - 1\} - (2\kappa/\pi)\, D_0 \sin(\pi x/2T_0) \quad (12)$$

$\int -y(x)\, dx =$

$[(D_0T_0(\kappa/\pi^{2} - 1/2) \sin(\pi x/T_0)$

$+ D_0(1/2 - \kappa/\pi)\, x$

$- (\beta T_0/2\pi)\{\sin(2\pi x/T_0) + \beta x$

$- (4\kappa/\pi^2)\, D_0T_0 \cos(\pi x/2T_0)\,]$

$$= D_0T_0(0.5 - \kappa/\pi + 4\kappa/\pi^2) + \beta T_0 \qquad (13)$$

The correction for the additional sample at $x = T_1$ becomes:

$\beta = \beta U / \beta L$
$\beta U =$
$- D_1 - D_0(1/2 - \kappa/\pi)\{\cos(\pi T_1/T_0) - 1\}$
$+ (2\kappa/\pi)\, D_0 \sin(\pi T_1/2T_0)$
$\beta L = \cos(2\pi T_1/T_0) - 1$

Estimating κ

One obvious way of determining a value for κ is to measure the slopes of the stream bed at the banks. However there two objections to this.

When the banks are steep, $dy/dx(0)$ becomes extremely large and since

$\kappa = - dy/dx(0)\, To/Do$
so therefore does κ.

Mathematically this creates problems as a near- infinite slope requires terms of near-infinite amplitude, and slight errors in their differences become exaggerated.[7] To avoid this problem, values of κ greater than 5 should not be used.[8]

The second problem is that measuring the slopes are measurements, and these we want to avoid.

Since we are dealing with a tweak, and the value of κ is, for reasons explained, not critical to an accurate estimate of the cross-sectional areas, an adequate substitute is to use a value of slope determined at T_1. Hence:

$\kappa = D_1To/DoT_1$
else $\kappa = 5$ if $D_1To/DoT_1 > 5$ (14)

This avoids both problems. It also makes κ a function of water level, which except in a draw, it always is.

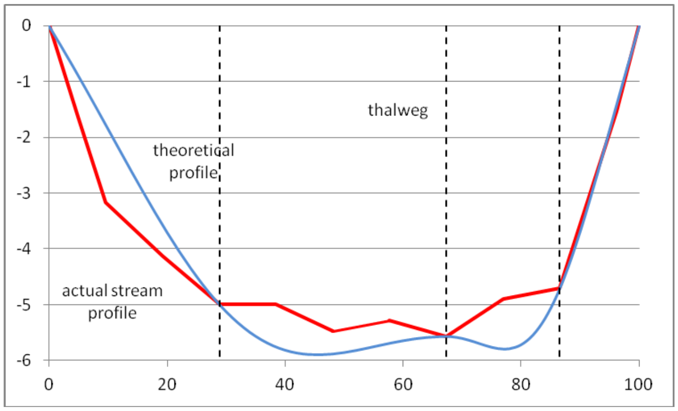


Figure 8: The real stream profile used in Fig.5 and the new and improved estimated profile. The cross-sectional area error has been reduced to +1.9%.

Reality checks

Figure 8 shows the result of applying Eq.13 and Eq.14 to a real stream profile. The error is cross-sectional area has been reduced from –35% to +1.9%. Perfection is of course not possible because without observation, we cannot know for sure what the stream bed does between the sampling points. All the equations are doing is making a guess with the assumption that there are no abrupt changes in stream depth.

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[7] It is possible to eliminate the error in the estimate of area for very large κ—it is of order $0.021\kappa D_0W_0$—but it's not worth it my opinion as although the area remains reasonable, the profile becomes nonsensical and calculations have to be made with great precision.

[8] Loosely this means don't use bank slopes greater than 5 times the average slope.

This will certainly be true if the stream is flowing over erodible or deposited soil; less so if the bed is hard rock or there are obstructions below the surface.

For a perfectly V-shaped profile:

A_L (T_1=T_0/2, D_1=D_0/2) = 0.52 D_0T_0

which is only slightly more than the Robison and Beschta value of 0.50 D_0T_0.

For the difficult case of a perfectly rectangular flat-bottomed stream, Eq.13 and Eq. 14 give an error in the estimated area of –20% compared to the correct value of D_0W. Dropping Eq.14 and using the maximum recommended value for κ of 5 reduces this to –14.6%. Still not perfect, but if the profile really is rectangular, which is unlikely if the course is a natural one, then we can dispense with the math and call the area $A = A_L + A_R = D_0W$ (15)

For a semi-circular profile, Eq.13 and Eq. 14 give an error in the estimated area of –4.1% compared to the correct value of $\pi W^2/8$. Dropping Eq.14 and using the maximum recommended value for κ of 5 actually increases the magnitude of the error to +12.8%. Another illustration of the difficulty of dealing with vertical banks; however, the only perfectly semi-circular profile one is likely to come across is in a culvert, and this is unlikely to be exactly half full so the more direct approach is what's needed:

$$A = A_L + A_R = R^2\{\mathrm{acos}(1–d) – (1–d)\sqrt{(2d–d^2)}\} \quad (16)$$

where
R = radius of culvert barrel
d = (water depth at the centre) /R

Velocity estimates

The problem of accurately measuring the velocity of the water flowing through a transverse cross-sectional area with as few as measurements as possible is, if anything, even more difficult than estimating the area. Velocity in the downstream direction is a function of stream depth, depth below the surface, distance from the bank, turbulence including eddies or the lack thereof, and the presence or absence of aquatic plants.

Let's start with:

$$Q_L = \int\int V(z,x)\, dz\, dx$$

where

Q_L is the volumetric flow (m^3/s) to the left of the thalweg, the process for the right side being identical;

V(z,x) is the downstream velocity at a point z above the stream bed and a distance x from the left bank;

the inner integral is from z = 0, the stream bed to z = –y(x) the stream surface; and

the outer integral is from x = 0 at the bank to x = T_0 at the thalweg.

The function V(z,x) is, for present purposes, an impossibly complicated function that varies from cross-section to cross-section and from stream to stream. However, one way of simplifying this somewhat, even if crudely, is to write:

$$Q_L = V_0 \int f(x)\, g(x)\, D(x)\, dx \quad (17)$$

where:
Q_L is the volumetric flow (m^3/s);

V_0 is the surface velocity (m/s) at the thalweg;

f(x) is a dimensionless function relating the surface velocity at x to the surface velocity at the thalweg;

g(x) is is a dimensionless function relating the average velocity of a vertical section at x to the surface velocity at x;

$D(x) = -y(x)$ (17a)
the stream depth (m) at x from Eq.13 and Eq.14; and

the integral (m^2) is from x = 0 the bank to $x = T_0$ the distance of the thalweg from the bank.

For convenience, I'll sometimes write the variable D(x) as just D, and as earlier, $D(T_0) = D_0$.

Hence we have:

$V_0 = V(D_0,T_0)$ (17b)

$f(x) = V(D,x) / V_0$ (17c)

with integration to find Q_L in Eq.17 from
x = 0 to T_0; and

$g(x) = \int V(hD,x)\, dh / V(D,x)$ (17d)

where V(hD,x) is the velocity at the relative height h above the stream bed (h = 1 at the surface, h=0 at the bed) at distance x with integration from h = 0 to h=1.

Some practical values for f(x)

Thanks to the gentleness of nature, both f(x) and g(x) are very amenable to being represented mathematically by 2nd-order polynomials. Looking at f(x) first, we can write:

$f(x) \approx \alpha_H x^2 + \beta_H x + \gamma_H$ (18)

with two boundary conditions
$f(0) = 0$ and $f(T_0) = 1$

that is, at the bank, the velocity falls to 0; and at the thalweg, the ratio of the surface velocity to the surface velocity at the thalweg is by definition 1.

Note: One concern with this is that f(x) is not just a function of x as implied by Eq.18, but is also a function of D, the stream depth. However for a "generic" stream there is a strong linear relationship between x and D as discovered by Robison and Beschta. As a result, f(x) is not very different from f(D). All my attempts to invent a function involving both x and D such as
$f(x,D) = \sqrt{\{f(x).f(D\}}$
have produced a worse estimate of Q_L than does Eq.18. I have not attempted a multi-variate analysis because I don't have a data bank, but I suspect this would, in any case, be a waste of time when we are looking for a solution for streams in general, not a particular stream. It appears that the other functions in the integration in Eq.17 deal adequately with the variation in D across the section, and f(x,D) can indeed be simplified to f(x) when used in combination with the other functions.

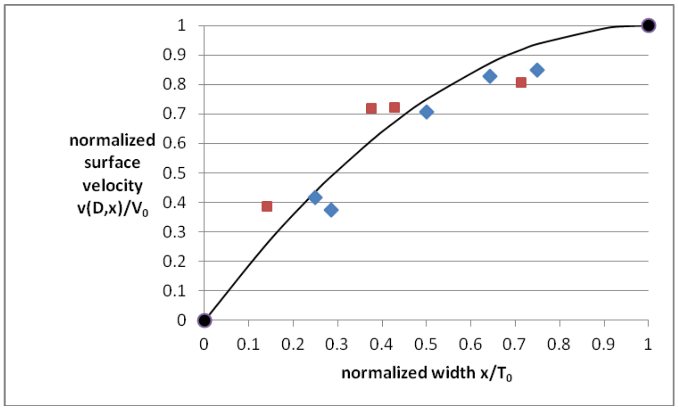

Figure 9: The surface velocity relative to the surface velocity at the thalweg as a function of normalized distance from the thalweg for two real streams (red and blue) on both sides (left and right). The curve is the prediction of Eq.18 and Eq.19a and matches closely the trendline for the data.

Note that forcing the theoretical curve to go through the data points at the approximately half-way position (T_1/T_0= 0.5) would not change the curve much, though other streams might be different.

The boundary conditions for Eq. 18 provide two of the three conditions needed to give numerical values to α_H, β_H, and γ_H, so we need a third. There's a choice. Either we can require:

$df(T_0)/dx = 0$ no change in surface velocity for short distances either side of the thalweg, in which case:

$\alpha_H = -1/T_0^2$
$\beta_H = 2/T_0$
$\gamma_H = 0$ (19a)

or, we could use:

$\alpha_H T_1^2 + {}_H T_1 = V_1/V_0$ (19b)

where V_1 is the surface velocity at T_1 from the bank, approximately midway between the bank and the thalweg. Measuring this using the float method would however cause a lot of problems in the field.

If we did use it, we would have:
$\gamma_H = 0$
$\alpha_H T_0^2 + \beta_H T_0 = 1$
$\alpha_H T_1^2 + \beta_H T_1 = V_1/V_0$ (19c)

which is sufficient to fix the values of the constants α_H, β_H, and γ_H. I won't clutter up the text here by including messy algebra that Figure 9 suggests we probably don't need.

Some practical values for g(x)

The function g(x) represents the ratio of the average flow of a column of water at x to the surface velocity at x.

There's lots of websites and text books that one way or another pay a lot of attention to determining the average velocity $\Phi(x)$ of a vertical section:

$\Phi(x) = \int V(hD(x), x)\, dh$

integration from h = 0 to 1

The four choices offered for determining $\Phi(x)$ are:

(i) $\Phi(x)$ = a flow meter measurement

(ii) $\Phi(x) = V(0.6D,x)$

(iii) $\Phi(x) =$
$0.5\{V(0.8D, x) + v(0.2D,x)\}$

(iv) $\Phi(x) = \lambda\, V(D,x)$

<u>Choice (i)</u>: With modern instruments and software, determining the average accurately can be accomplished by moving a flow meter sensor at a steady pace from the surface to the bed and using its internal software to produce an average. But such equipment is expensive. I'll assume it's not available.

<u>Choice (ii)</u> says that the average velocity of a vertical section is the same as the velocity at a depth 0.6 times the stream depth, the velocity at the surface being V(D,x). This is a rather crude estimate, and I think we can do better than that.

<u>Choice (iii)</u> says the average velocity is the average of the velocities at depths 0.2 and 0.8 of the stream depth. This is, or was, a USGS standard; it is a bit old fashioned nowadays. In my limited experience this does appear to be true, but who wants to go calculating what the 0.2 and 0.8 stream depths are, and then measuring the velocity at precisely those depths. It involves a lot of wading and measuring if you do this all the way across the stream.

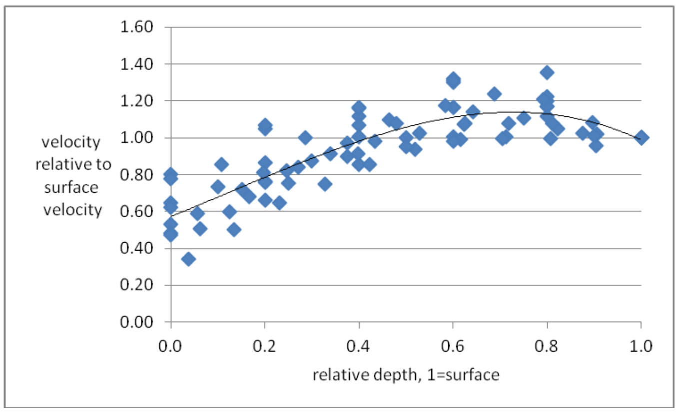

Figure 10: The variation of velocity at depth compared to the velocity at the surface for three real creeks. The spread is partly due to the points being at different distances from the bank. Note the non-zero velocity at the stream bed, an indication that the flow there is turbulent.

Choice (iv), the one I'm proposing to use, says that the average velocity of a vertical section of the stream is a quasi-constant λ times the surface velocity. Popular choices for λ, treating it as a constant, are $0.8 < \lambda < 0.9$. Unfortunately, it is only approximately true that λ is a constant. It depends on whether the stream flow is laminar or turbulent, and the proximity of the measurement to the bank.

In all but very turbulent creeks, the highest velocity of the water occurs below the surface. When the flow is laminar, the velocity falls from its maximum to near zero at the bottom, $V(0,x) = 0$. However, when the flow is turbulent, the bottom layer is mixed with water higher up in the column, and this mixing raises the average stream bed velocity above zero, $V(0,x) > 0$. You can see this effect in Figure 10.

For these particular creeks, the average velocity with depth compared to the surface velocity, as indicated by the trend line in Figure 10, is 0.96 times the surface velocity, and we could use this for all distances from the bank for these creeks.[9] However, in general, although the average velocity of the water column is always less than the surface velocity at the thalweg, $\lambda<1$, the depth below the surface of the velocity maximum sinks as you move towards the bank and the water shallows. Away from the thalweg, the average velocity of the column rises toward the surface velocity and may even surpass the surface velocity, $\lambda > 1$. At the banks however, the average and surface velocities converge to the same value as the water column thins to nothing at the shore, $\lambda = 1$.[10]

When looking at the function

$g(x) = \int V(hD(x), x)\, dh / V(D(x), x)$

The same concern arises that arose when thinking about f(x). The function g(x) is not just a function of x, but is also a function of D, the stream depth. In the case of f(x), an argument was made that nevertheless, for practical purposes we can regard f(x) as a function only of x, the distance from the bank. In the case of g(x), I'm going to make the opposite choice, namely, to consider g(x) as a function of D rather than x.

Note: The argument for this goes as follows. The end result we are looking for is an accurate assessment of volumetric flow, not of velocity *per se*. We are therefore more interested in getting the velocity right near the thalweg than we are in getting it right in the shallower water because the shallower water near the banks contributes significantly less to the volumetric flow than does the water near the thalweg. The thalweg is on average the most distant point from the banks, and therefore one can surmise that any variation in velocity in the vicinity of the thalweg is going to be more influenced by variation in the depth than it is by variation in the distance from the banks. To add to this, we can note that in natural streams there is in any case a strong correlation between x and D, so, on average, it does not matter much which variable we use.

Looking at g(x), we can write:

$$g(x) \approx \alpha_V (D/D_0)^2 + \beta_V (D/D_0) + \gamma_V \qquad (20)$$

with two boundary conditions
$g(0) = 1$ and $g(T_0) = \lambda_0$

[9] A value of $\lambda > 0.9$ usually indicates a smoothly flowing stream with a muddy or bedrock bottom. However, such streams also have, in theory, a near zero velocity at the stream bed, but the data does not always support that, possibly because of measurement problems.

[10] There's good discussion of this in Marie Morisawa, *Streams—their dynamics and morphology*, Chap.3, *The hydraulics of streams*, McGraw-Hill, 1968.

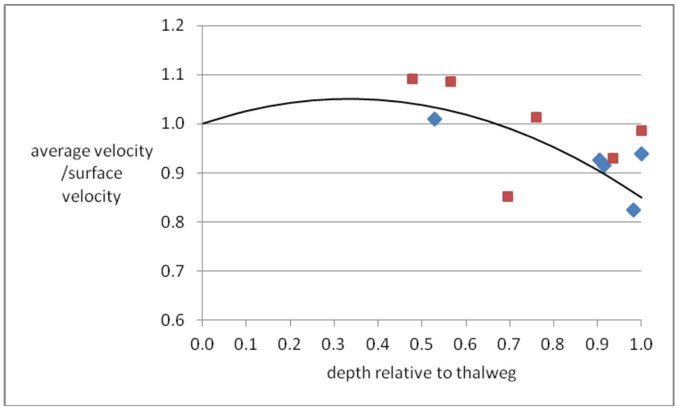

Figure 11: The average velocity relative to the surface velocity versus the stream depth relative to the thalweg depth for two real streams (red and blue) on both sides (left and right). The curve is the prediction of Eq.20 and Eq.21 for a proposed "generic" stream.

Note the relative velocity scale is not the full range 0 to 1.

where λ_0 is the ratio of the average velocity to the surface velocity at the thalweg (T_0).

The boundary conditions for Eq.20 provide two of the three conditions needed to give numerical values to α_V, β_V, and γ_V, so we need a third. My vote would be for a limit on the maximum value of λ. Values of λ_{max} around 1.05 appear, looking at my limited available stream data, to be common.

Hence we have:

$$\gamma_V = 1$$
$$\alpha_V + \beta_V + \gamma_V = \lambda_0$$
$$\beta_V = \sqrt{\{4\alpha_V(1 - \lambda_{max})\}}$$

This gives us:

$$\alpha_V = (1+\lambda_0-2\lambda_{max}) - 2\sqrt{\{\lambda_0-\lambda_0\lambda_{max}-\lambda_{max}+\lambda_{max}^2\}}$$
$$\beta_V = \lambda_0 - (1 + \alpha_V)$$
$$\gamma_V = 1.0 \qquad (21)$$

According to the voluminous literature on the topic, a commonly used default value for λ_0 is 0.85, which gives us $\alpha_V = -0.45$, $\beta_V = 0.30$, and $\gamma_V = 1.0$.

Measuring volume

The method thus described hinges on only one measurement of velocity that of V_0, the surface velocity at the thalweg. Whilst this minimizes the amount of field data that has to be gathered—it could hardly be less as far as velocity is concerned[11]—it does mean that the accuracy of the volumetric measurement will depend directly on the accuracy with which this single parameter is measured. How to do this, I'll leave to another discussion; however, I will note that attempting to measure velocity at the thalweg by timing the drift of a float over a measured reach of the stream has not been a satisfactory experience for me.[12] This is because you can't make the float follow the line of highest surface velocity; there are often eddies that move the float in directions other than directly downstream; and hidden underwater obstructions can drastically alter flow velocities locally.

Volumetric estimates

Does it work?

To test the method, I took an example of standard stream data from a surveyor's handbook.[13] The data is shown graphically in Figure 12. The FR is 19:1. Numerical

[11] Though when I was doing a groundwater study on Gabriola and was estimating run-off in streams, I found, after a bit of practice, I could make a not-too-bad guess as to what the velocity was just from the noise the stream made!

[12] More than once I've had a float get trapped in the void behind a pourover (submerged boulder or ledge) and remain there until retrieved. The average stream velocity there was zero!

[13] Charles B. Breed & George L. Hosmer, *The principles and practice of surveying*, Vol.2, *Higher surveying*, p.542, Edition. 7, Wiley and Sons, 1953.

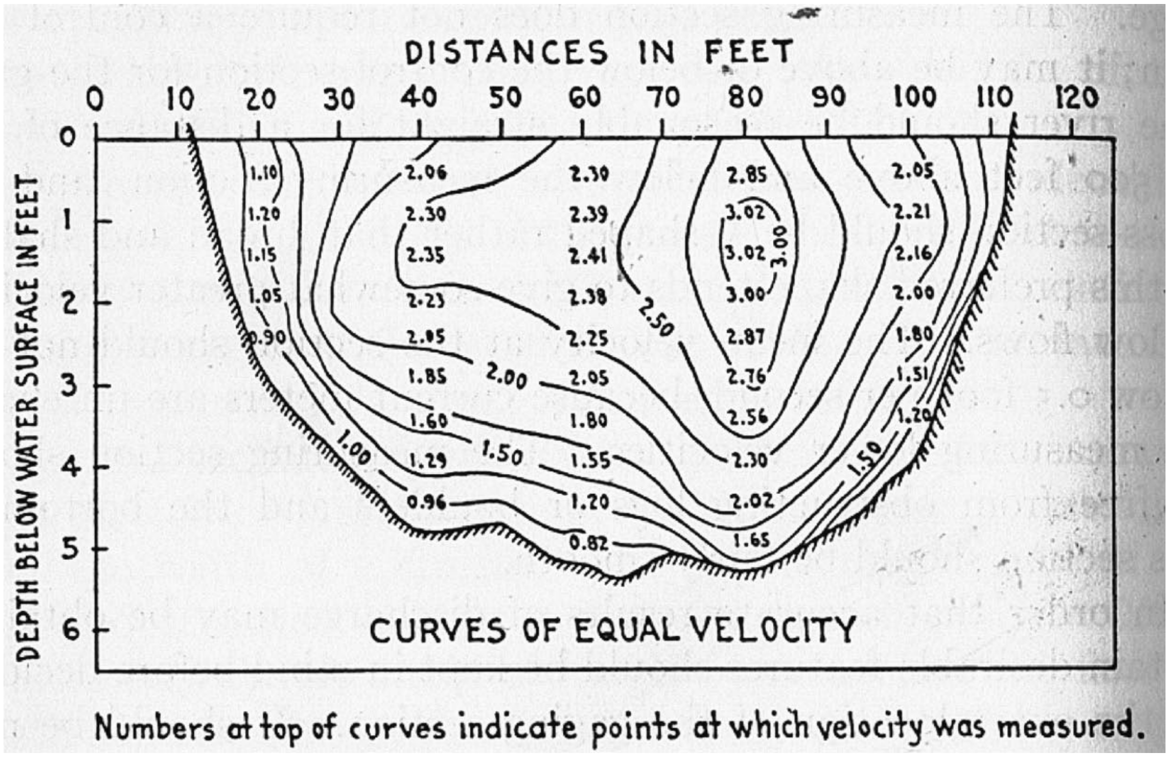

Figure 12: The profile and velocity matrix of a creek measured in the traditional way.

integration (without interpolation) of the data in this figure by textbook methods, and numerical integration of Eq.17 gives the following results:

Flow by standard: 815 cu.ft./s
Flow by equation: 773 cu.ft./s (–5.1%)

Area by standard: 411.9 sq.ft
Area by equation: 410.0 sq.ft (–0.4%)

Not bad considering that the surveyor made about 40 velocity measurements and about 60 x:y co-ordinate determinations, whilst solving Eq.17 requires only one velocity measurement and six co-ordinate measurements.[14]

In another example for a different creek[15] with FR 15:1, I obtained:

Flow by standard: 353 cu.ft./s
Flow by equation: 358 cu.ft./s (+1.6%)

Area by standard: 201.8 sq.ft
Area by equation: 194.3 sq.ft (–3.7%)

Again not bad.[16] There were about 36 velocity measurements and about 42 x:y co-ordinate determinations in the original data.

The most significant weakness of the analytical method presented here, assuming that equally good results can be obtained for other creeks, is in the choice of the value λ_0. A difference of the used value from the actual value contributes directly to the error in volumetric flow estimation.

That's something to look at when it comes to choosing a way of measuring velocity.

[14] For no velocity error, but retaining the area error, λ_0 should have been 0.90, not 0.85.

[15] Breed & Hosmer (ibid), p.555.

[16] For no velocity error, but retaining the area error, λ_0 should have been 0.78, not 0.85.

Culverts

The area estimation did not work for culverts and the same is true for velocity. For a circular culvert with no embedment we had:

$$A = A_L + A_R = \eta R^2 \qquad (16)$$

where
R = radius of culvert barrel
d = (water depth at the centre) /R
$\eta = \mathrm{acos}(1–d) – (1–d)\sqrt{(2d–d^2)}$

For velocity we can use the empirical equation:[17]

$$\lambda_0 = 1/\{(0.047–0.029B)T^{(–2.7+3.6B)} +1.38 \} \qquad (22)$$

where

λ_0 = ratio of the average velocity to the velocity at the surface at the mid-point of the culvert outflow[18]

$B = d\ \delta$
$T = \eta\ \delta^2$
$\delta = 2\sqrt{(2d–d^2)}/ \eta$.

For example, for a quarter-full culvert of any size, we have:
$d=0.5$, $\eta = 0.614$, $\delta= 2.820$, $B=1.41$, $T= 4.885$, and $\lambda_0 = 0.608$.

And the volumetric flow, Q, is:

$$Q = \eta R^2 \lambda_0 V_0 \qquad (23)$$

◊

[17] US Department of Transportation, *Fish passage in large culverts with low flows*, August 2014. This is Equation 73 in this very informative publication.

[18] If the outflow is perched, the maximum velocity at the surface has to be observed before the water leaving the culvert has gained velocity as it begins to fall.