Divvying up the pie (or pizza)

Nick Doe

At a dinner party I was once at, we had for desert, a fruit pie. There were to be eight guests, but one of us could not make it, so the task arose of dividing the pie by seven. It was a very delicious looking pie, so all present insisted that the portions be equal.

Being a know-it-all engineer, I boldly announced that we were in luck, because it is possible to divide a circular pie exactly into seven using only simple tools—meaning a compass and a straight edge, things that every good dinner host has to hand.

However, I was wrong. You can divide a pie (or pizza) exactly by all the integers up to twelve, using only a compass and straight edge, except for seven, nine, and eleven. Any exact method for these integers has an infinite number of steps—it is, in other words, too complicated, even for an engineer, to use at the dinner table. There are however some very good approximate methods of dividing by seven, nine, and eleven, and these are sometimes claimed, albeit falsely, to be exact.

In the case of seven, 360º divided by seven is 51.42857º…, which is not very promising as you couldn't even do that with a protractor. Dividing by eleven is similarly difficult, 360º divided by eleven is 32.72727º…

However, dividing 360º by nine is 40º, and you would think that you could do that by dividing by three and dividing each slice again by three; however, conventional wisdom has it that you cannot divide angles by three, and as I discovered in writing up these notes, it's true, you can't.[1]

To spare myself future embarrassment, I prepared the following cheat sheets which I take with me whenever I'm invited out to dinner and the number of guests present likely wanting a portion of pie (or pizza) is not known in advance. You may complain that it doesn't cover the case where there are thirteen guests. My first response is divide it by fourteen and then split one slice by thirteen…the extra slices would be so thin that not even an engineer would care about exactness. But wait a minute! If you could divide the pie (or pizza) by fourteen, you could also divide it by seven, which I just said, you can't. I think you'll need another website. Either that or one fewer guests.

Some ideas in these notes are from the very readable book, Michael S. Schneider, *A Beginner's Guide to Constructing the Universe*, Harper Perennial, 1995. I also had a bit of help from John Harris, *Some propositions in geometry—in five parts*, Wertheimer & Lea, London, April 1884. And Wikipedia is, as always, an essential read. Check out *Angle trisection* to see what I mean.

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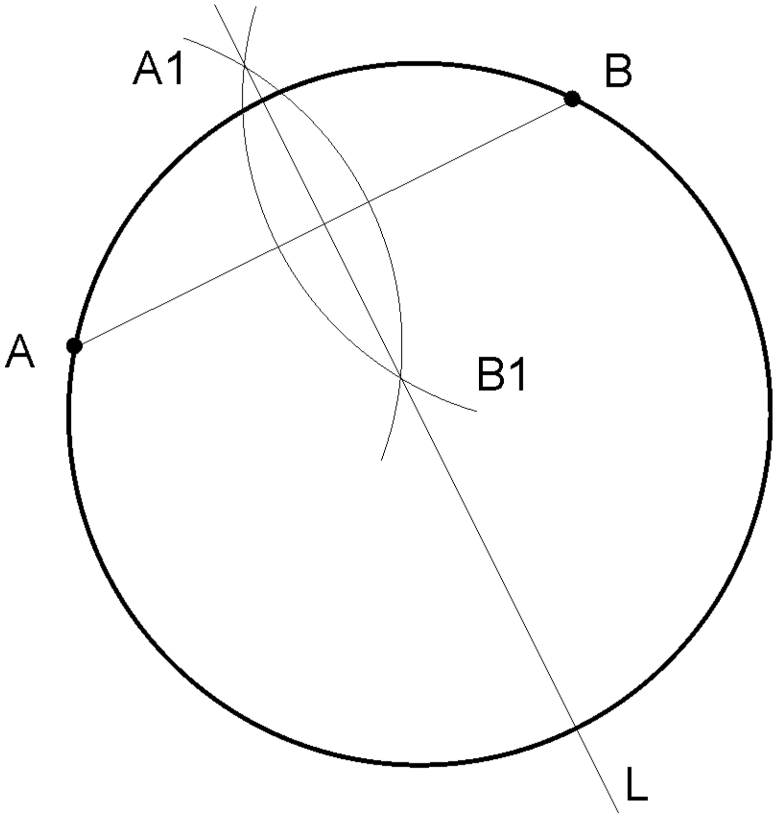
[1] In general you can't, but there are a few isolated exceptions, 90º being an obvious one. Less obvious is division of (15/7)36º = 77.14286…º by three, but it's simple. Just walk it round the circle five times leaving you with (75/7)36º, which is (70/7)36º, a full circle, plus (5/7)36º, which is exactly a third of the angle (15/7)36º.

TWO

Divide a circle into 2 equal parts using only a straight-edge and compass.



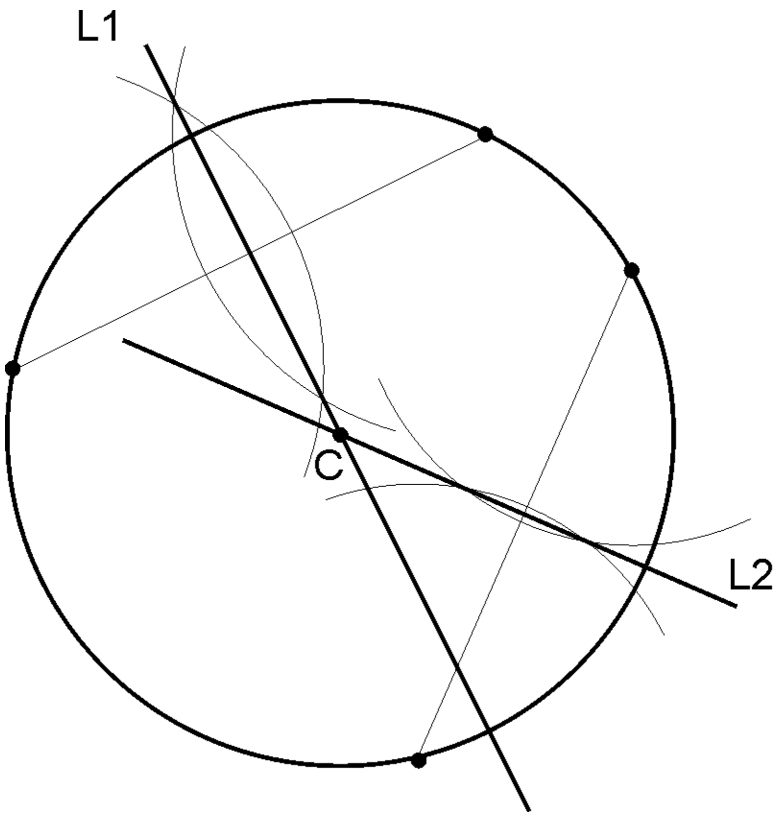
1. Draw a line AB between any two points on the edge of the circle.
2. With compass centre A at any convenient setting greater than a half AB, draw arc A1.
3. With compass centre B with the same setting used to create A1, draw arc B1.
4. A line L through the intersections of arc A1 and arc B1 will divide the circle in two.

Why does it work?

To divide the circle in two, you need a line through the centre. The centre is the point in the circle that is equidistant from all points on the edge. Hence a line that is equidistant from any two arbitrarily-chosen points on the edge cannot fail to go through the centre.

CENTRE

Find the centre of a circle.



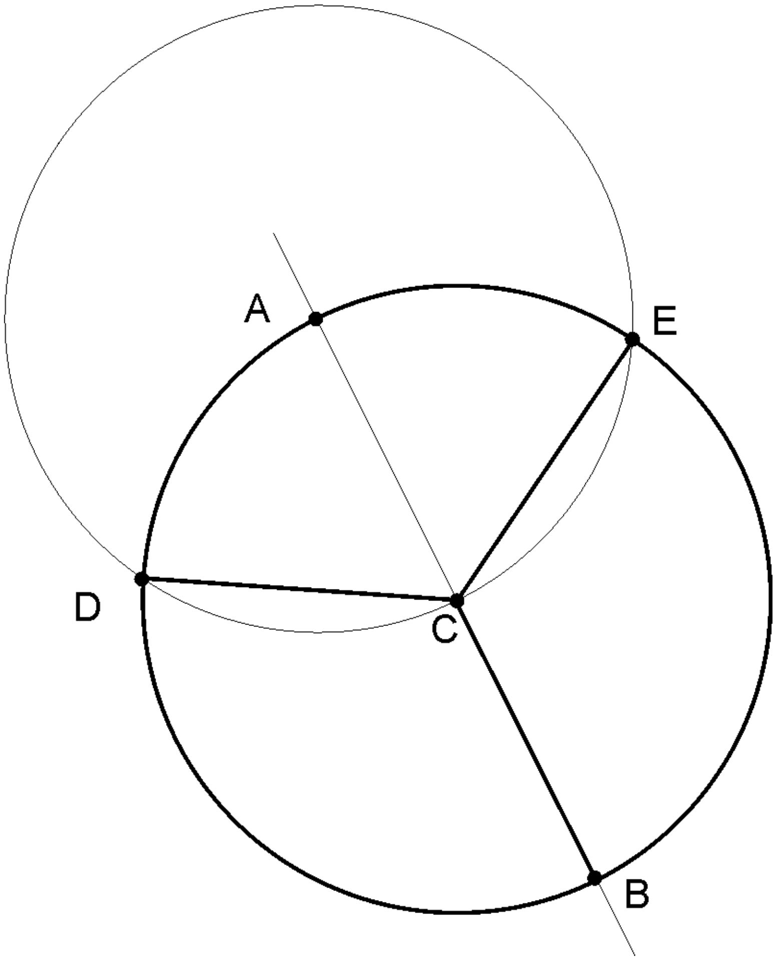
1. Divide the circle in two creating L1.
2. Select any different chord and again divide the circle in two creating L2.
3. The intersection of L1 and L2 will be the centre C.

Why does it work?

All diameters pass through the centre.

THREE

Divide a circle into 3 equal parts using only a straight-edge and compass.



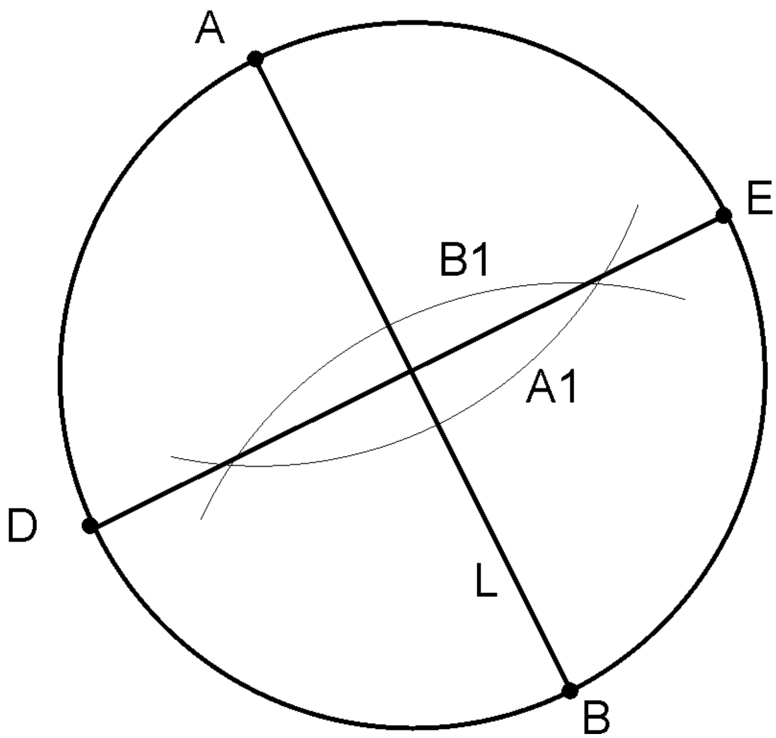
1. Divide the circle in two with the line AB.
2. Draw a circle, centre A, with radius CA.
3. The lines CD, CE, and CB divide the circle in three.

Why does it work?

Triangles CAE and CAD are equilateral with internal angles of 60º.
Therefore angle DCE is 120º
Similarly, angles DCB and BCE are also 120º.

FOUR

Divide a circle into 4 equal parts using only a straight-edge and compass.



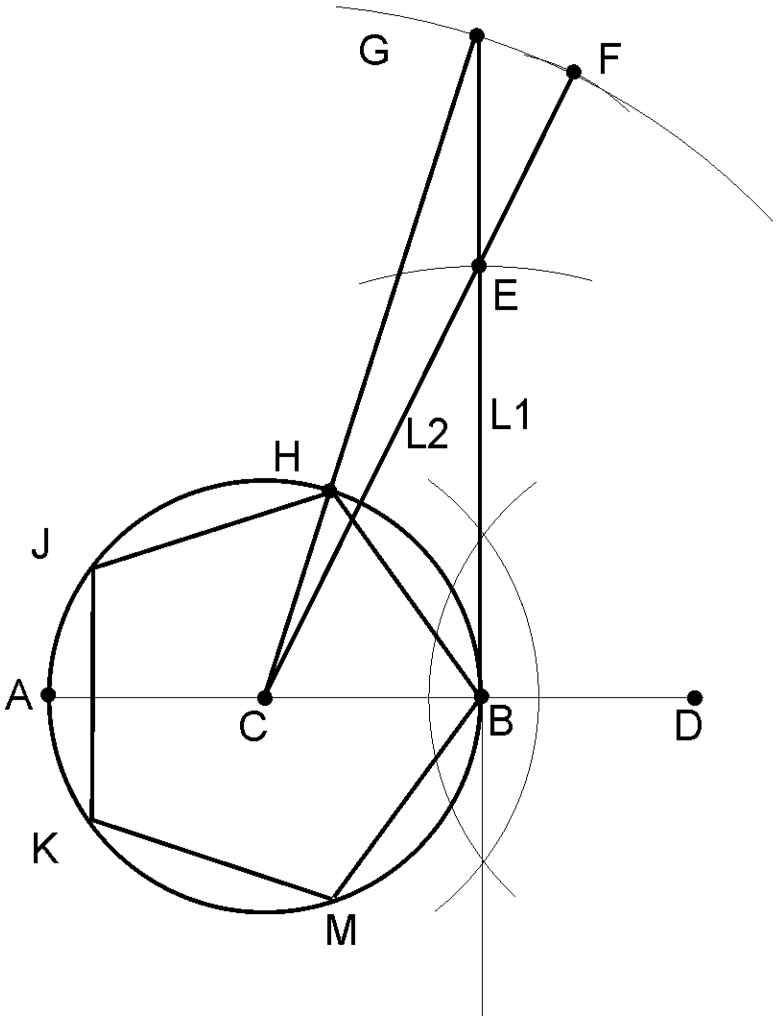
1. Divide the circle in two by creating AB, which is line L.
2. With compass centre A at any convenient setting greater than a half AB, draw arc A1.
3. With compass centre B with the same setting used to create A1, draw arc B1.
4. A line through the intersections of arc A1 and arc B1 to D and E will divide the circle in four.

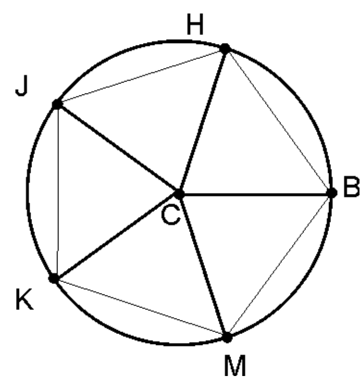
Why does it work?

A line equidistant from the ends of L must be at right angles to it and must divide it into two equal halves.

FIVE

Divide a circle into 5 equal parts using only a straight-edge and compass. Difficult but not impossible.





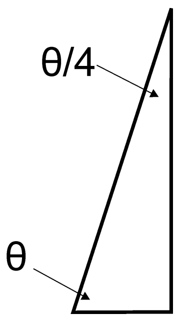
1. Draw the circle diameter ACB.
2. Extend the diameter ACB to D, making BD equal to the radius of the circle CB.
3. Draw arcs centre C and centre D and use the intersections to create line L1 through B.
4. Locate E on L1 at a distance from B equal to the diameter of the circle AB.
5. Join C to E to create line L2 and extend it one circle radius CB beyond E to F.
6. With compass centre C and set to CF, find the intersection with L1 at G.
7. The intersection of the line CG with the circle gives the second point of the pentagon at H.
8. With compass set to BH, draw chords from H to J to K to M to B, and back to H.
9. The points B, H, J, K, and M divide the circle exactly into 5.

Why does it work?

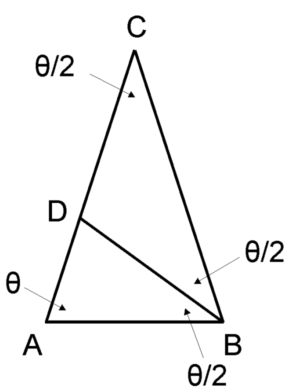
We need to create an angle of $360^\circ/5 = 72^\circ$.
We can do this by creating a right-angled triangle with its smallest angle = $90^\circ - 72^\circ = 18^\circ$. It just so happens that $\sin(18^\circ) = 1/(\sqrt{5} + 1)$, I'll explain why in a second.

The angle CGB is 18º because sine that angle = CB/CG; and.
CB is the radius of the circle, let's just call that 1 unit; and
CG = CE + EF = CE + 1; and
CE = $\sqrt{(CB^2 + BE)^2}$; and, since
CB = 1 and BE = 2, CE = $\sqrt{5}$; and hence
CG = $\sqrt{5}$ + 1; and so
CB/CG = 1/($\sqrt{5}$ +1).

Now, why is sin(18º) = 1/($\sqrt{5}$ +1) ? Here's an algebraic and a geometric answer.



In the right-angled triangle with angles θ and θ/4, θ must equal 72º
But we also have:
cos(θ) = sin (θ/4)
Algebraically, we have the textbook relationship, $\cos(\theta) = 8 \sin^4(\theta/4) - 8\sin^2(\theta/4) + 1$
So sin(θ/4) must be a solution of:
$8 \sin^4(\theta/4) - 8\sin^2(\theta/4) - \sin(\theta/4) + 1 = 0$
and believe or not, that is sin(θ/4) = sin(18º) = 1/($\sqrt{5}$ +1).



Or, if you don't like mysterious algebra, try this.

Construct the isosceles triangle ABC with the smaller angle at C equal to half the two equal angles at A and B.
Add the line BD such that the triangle ABD is also an isosceles triangle.
Call AC = BC = a; and call AB = 2b.

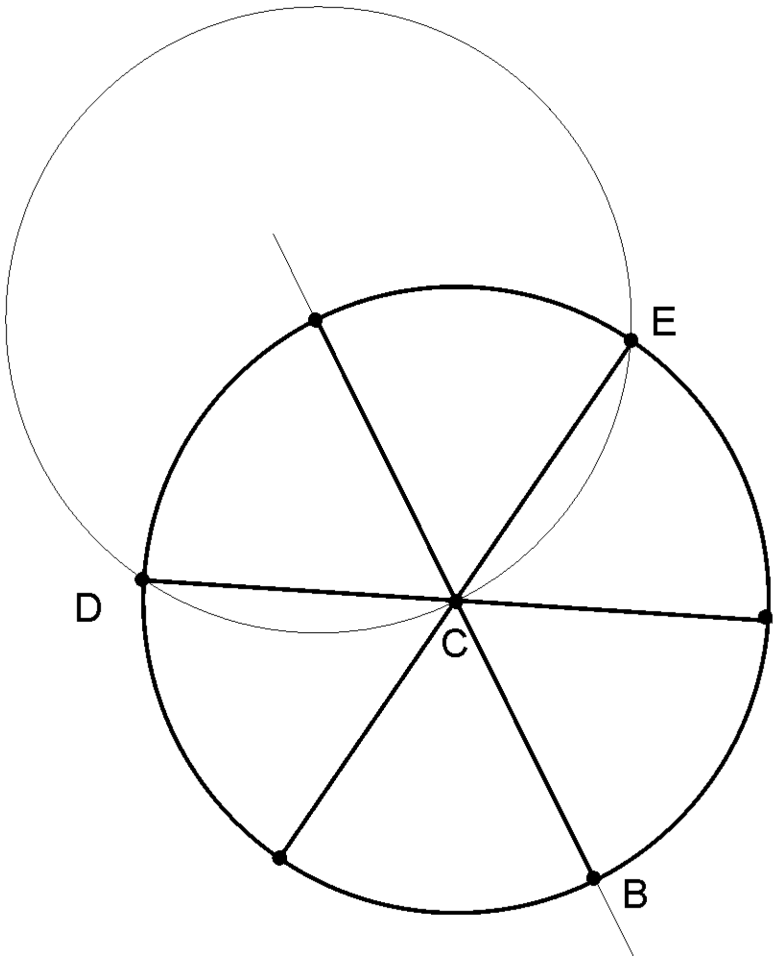
Then looking at half of angle ACB we have sin(θ/4) = b/a.
And looking at half of angle ABD we have sin(θ/4) = (DA/2)/AB.
But AB = 2b; and
DA = CA – DC, DC = DB = AB, and so DA = a – 2b; and so
sin(θ/4) = (a – 2b)/4b = a/4b – 1/2.
So b/a = a/4b – 1/2. Now, multipling by 4a/b leads to:
$(a/b)^2 - 2(a/b) - 4 = 0$.

OK, we do need a little bit of algebra to solve this quadratic equation:
(a/b) = 1 ± $\sqrt{(1 + 4)}$
Hence, sin(18º) = sin(θ/4) = 1/(a/b) = 1/(1 + $\sqrt{5}$).

There's probably a more elegant proof, but that's the best I know.

SIX

Divide a circle into 6 equal parts using only a straight-edge and compass.



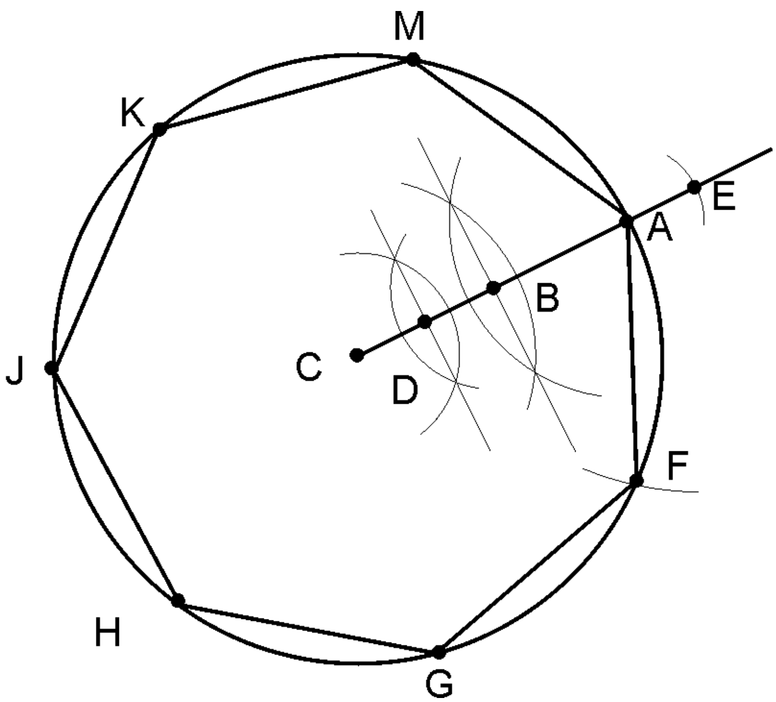
1. Divide the circle in three and extend the cuts.

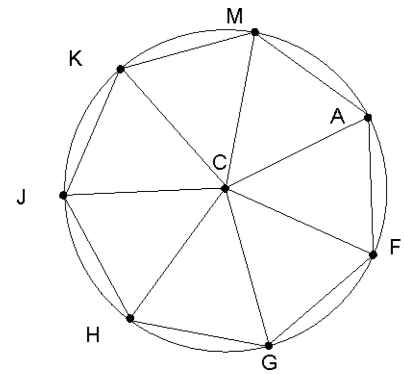
Why does it work?

It divides all the 120º angles in half.

SEVEN

Divide a circle into 7 equal parts using only a straight-edge and compass. This is not possible, but here's one of several good approximations.



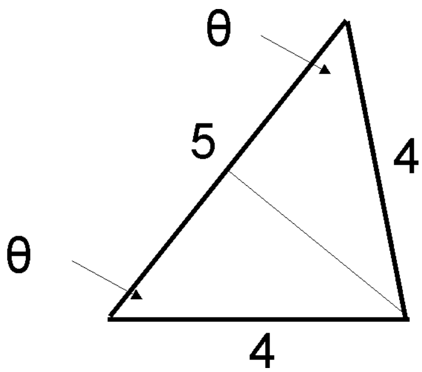


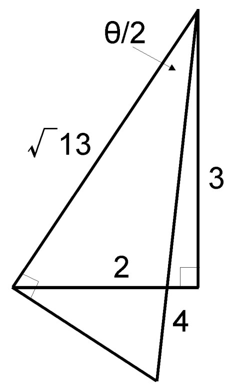
1. Draw the radius CA and extend it outside the circle.
2. Draw two arcs centre C and centre A; put a line through their intersections to find B.
3. Draw two arcs centre C and centre B, put a line through their intersections to find D.
4. With compass set to CD, centre A, locate E.
5. With compass set to CA (the radius of the circle), centre E, locate F.
6. Draw chord AF.
7. With compass set to AF draw chords from F to G to H to J to K to M, and close MA.
8. The points A, F, G, H, J, K, and M divide the circle into 7.

Why does it work?

We need to create an angle of $360^{\circ}/7 = 51.42857\ldots^{\circ}$

We may not be able to do this exactly, but we get very close using a 4:4:5 triangle. In ancient times, people used a string with 13 equally spaced knots to construct such triangles.

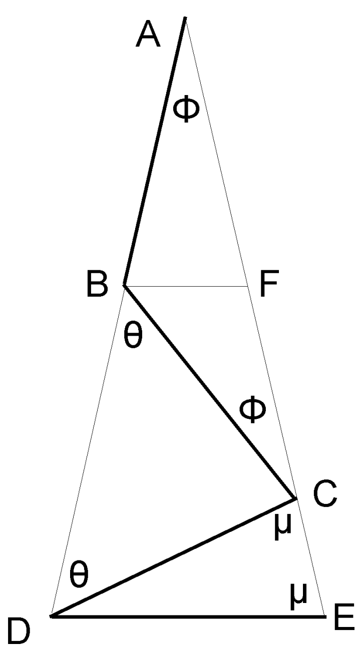




In an isosceles triangle with 4:4:5 sides and two internal angles θ, we have:
$2\ \{4\cos(\theta)\} = 5$
Hence: $\cos(\theta) = 5/8$ and $\theta = 51.31781\ldots^{\circ}$

This is only 0.11° short of what we want.. In the construction, CFE is a 4:4:5 triangle.
The greatest error will be in the final step as MA = $360^{\circ} - 6\ \mathrm{acos}(5/8) = 52.09312\ldots$, an error of $+0.7^{\circ}$.
You can probably do something similar using $\cos(\theta/2) = \sqrt{13}/4$ together with a $2:3:\sqrt{13}$ triangle.

A method of dividing a circle by 14, and hence by also 7, is the seven-toothpick method, much favoured by dinner guests. The diagram on the left shows the arrangement with 4 toothpicks, the other three being just horizontally-flipped mirror images of AB, BC, and CD.



Triangles ABC, BCD, and CDE are all isosceles.

For ABD to be a straight line we require:
$\theta + (180^{\circ} - 2\Phi) = 180^{\circ}$
Hence $\theta = 2\Phi$

For ACE to be a straight line we require:
$\Phi + (180^{\circ} - 2\theta) + \mu = 180^{\circ}$
Hence $\mu = 2\theta - \Phi = 3\Phi$

Now equating angle ABF and angle ADE, we have
$(180^{\circ} - \Phi)/2 = \theta + (180^{\circ} - 2\mu) = 180^{\circ} - 4\Phi$
Hence $\Phi = 90^{\circ}/3.5 = 360^{\circ}/14$

So why can't you use this to exactly divide a circle by seven? Because it involves never-ending successive approximation. The ratio of the toothpick lengths (AB, BC, CD, DE) to AD has to be 0.44504187… and it's difficult to trim toothpicks to exactly that, though 4/9 = 0.444… is a good start. Eyeballing instead to make ABD and ACE straight lines is however easy, even though in theory you need to extend AB and BD, and AC and CE to infinity in zero gravity to show that exactly.

All this doesn't really explain why $\cos(\theta) = 5/8$ is related so closely to division by 7, and I don't know the answer to that one, though I have tried to find one as you can see from the following notes.

<u>The algebraic approach</u> starts with the textbook relationship:
$64 \cos^7 A - 112 \cos^5 A + 56 \cos^3 A - 7 \cos A = \cos 7A$

If A =360º/7, then this becomes the degree-7, trignometric polynomial:
$64 \cos^7 A - 112 \cos^5 A + 56 \cos^3 A - 7 \cos A - 1 = 0$

This formidable-looking equation is not as bad as it looks. Although it has seven roots corresponding to $A(n) = 360º \times n/7$, $n = 0$ to 6, one is a simple root, $\cos A(0) = 1$, and the other six roots are three double roots because $\cos A(n) = \cos A(7-n)$, which makes $\cos A(n)$, $n = 4,5,6$ the same as $\cos A(n)$, $n = 3,2,1$.

The equation can thus be written:
$(\cos A-1)(8\cos^3 A + 4\cos^2 A - 4\cos A - 1)^2 = 0$

which leaves us with the problem of solving the cubic equation:
$8\cos^3 A + 4\cos^2 A - 4\cos A - 1 = 0$

Cubic equations are solvable, but not easily. The methods involve complex numbers even when all the roots are real. A simpler approach is to factorize the equation thus:
$(8\cos A - 4) (\cos^2 A + \cos A) = 1$

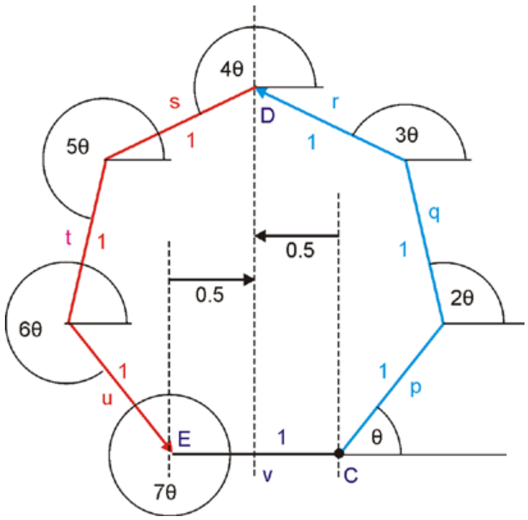
Now, if $(8\cos A - 4) > 1$, that is, $\cos A > 5/8$, then by inspection, $(\cos^2 A + \cos A)$ must be < 1.
Solving the quadratic equation $\cos^2 A + \cos A - 1 = 0$, gives us $\cos A = (\sqrt{5} - 1)/2$.

Hence, if $\cos A > 5/8$ [>0.625], then cosA must also be $< (\sqrt{5} - 1)/2$ [<0.618], which is not possible. This means for cosA to be a solution, it must meet the restrictions:
$(\sqrt{5} - 1)/2 \leq \cos A \leq 5/8$.

The rational numbers that meet these restrictions are:
cosA = 5/8, 28/45, 33/53, 38/61, etc.

of which, cosA = 5/8 is most useful. To be an order of magnitude better, you have to go to 48/77, and for two orders better, 154/247.

This still doesn't answer the question as to why cosA = 5/8 is such a good approximate solution; it appears from this to be just a co-incidence.



The geometric approach is different, but gets to the same place.
Starting at C (0,0) on the heptagon, we have the three vectors p, q, and r, which brings us to D, giving us:
cosA + cos2A + cos3A = –0.5

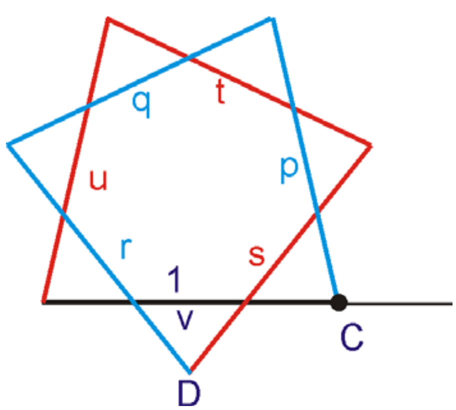
The mirror image, s, t, and u, similarly gives us:
cos4A + cos5A + cos6A = –0.5

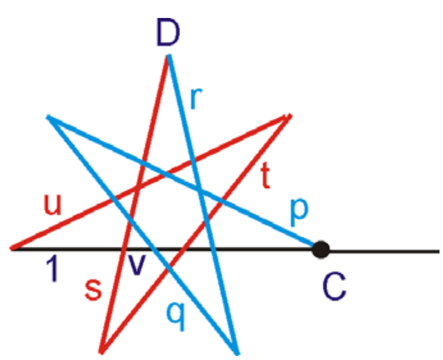
Hence, since:
cos4A = cos3A; cos5A = cos2A; and cos6A = cosA
2(cosA + cos2A + cos3A) + 1 = 0
which using, cos2A = $2\cos^2$A – 1; cos3A = $4\cos^3$A – 3cosA; gives us the algebraic solution:
$8\cos^3$A + $4\cos^2$A – 4cosA – 1 = 0.

The algebra is complicated because there are other figures besides a heptagon that fit this relationship. Using multiples of 2A and 3A instead of A, for example, gives us:

cos2A + cos4A + cos6A = –0.5
cos8A + cos10A + cos12A = –0.5; hence
cosA + cos3A + cos5A = –0.5

cos3A + cos6A + cos9A = –0.5; hence
cos3A + cos6A + cos2A = –0.5
cos12A + cos15A + cos18A = –0.5; hence
cos5A + cosA + cos4A = –0.5

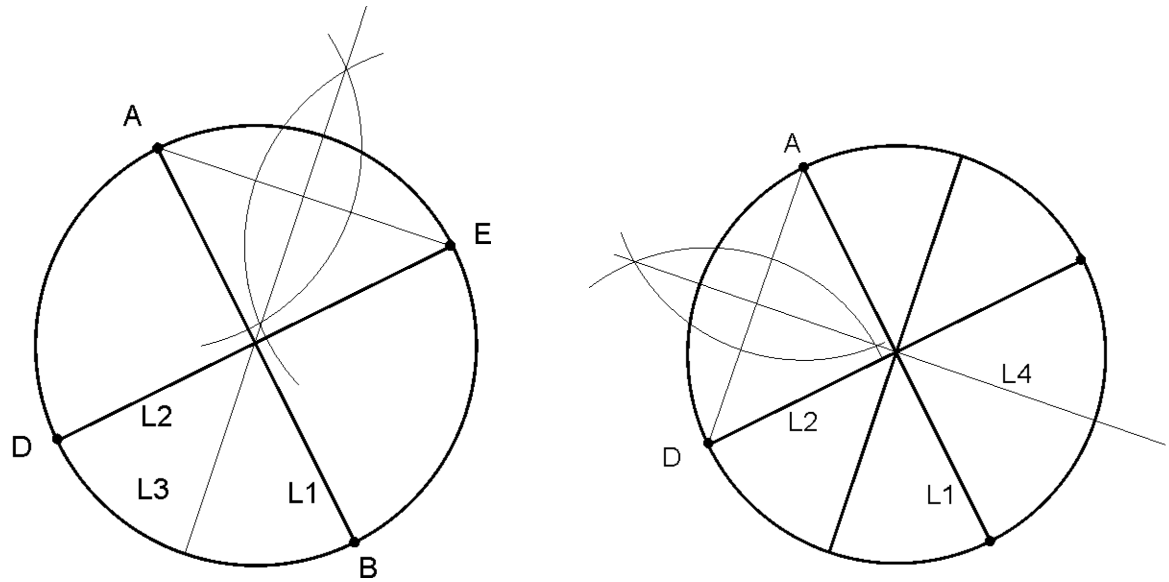




Higher multiples of A produce figures that are heptagons or are the same as these, or are rotated versions of these. To cope with this algebraically, you need to use complex (two-dimensional) numbers, but I doubt that would be more illuminating than is the geometry shown.

EIGHT

Divide a circle into 8 equal parts using only a straight-edge and compass.



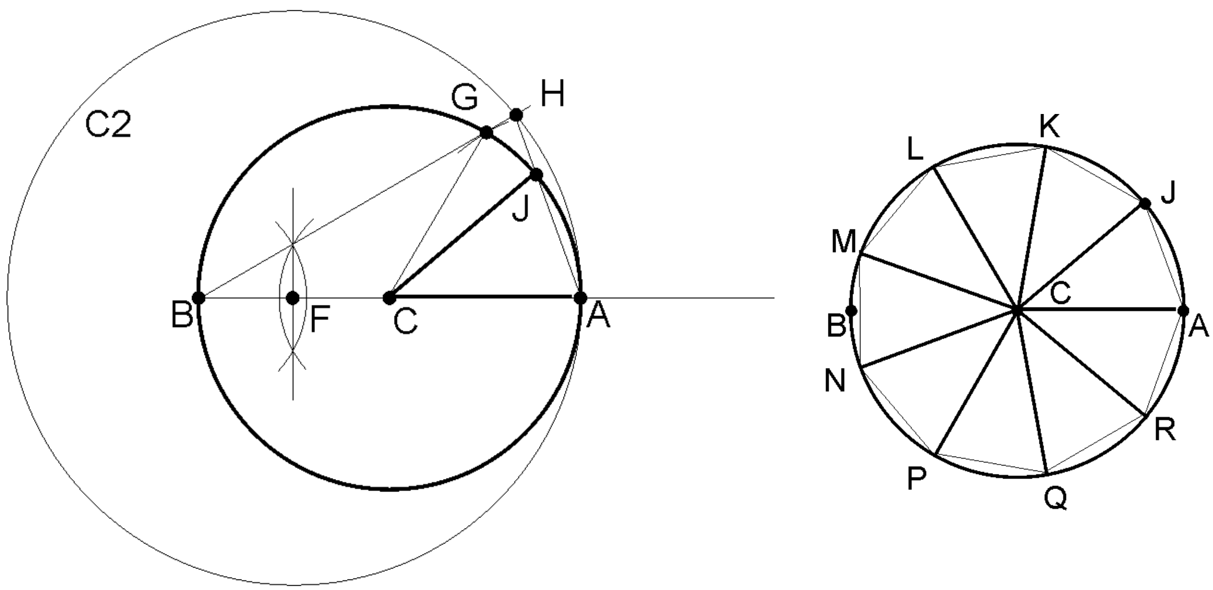
1. Divide the circle into four by creating L1 and L2.
2. With compass centre A at any convenient setting greater than a half AE, draw arc.
3. With compass centre E with the same setting, draw another arc.
4. A line L3 through the intersections of the arcs will divide the two quadrants.
5. Construct L4 bisecting chord AD to divide the other two quadrants.

Why does it work?

A line equidistant from the ends of a line must be at right angles to it and must divide it into two equal halves.

NINE

Divide a circle into 9 equal parts using only a straight-edge and compass. This method is approximate.



1. Draw the diameter of the circle B through C to A.
2. With compass set to more than half BC, draw arcs centre C and centre B and put a line through their intersections to locate F half way between B and C.
3. Draw a second circle C2, compass set to FA, centre F.
4. With compass set to CA, centre A, locate G on the circle.
5. Project B through G to H on circle C2.
6. Join H to A to locate J on the circle to be divided.
7. With compass set to chord JG, centre G, locate K; with compass set to AJ centre K, locate L; with compass set to AJ, centre L, locate M; with compass set to JK, centre M, locate N; with compass set to AJ, centre N, locate P; with compass set to AJ, centre P, locate Q; with compass set to JK, centre Q, locate R; close AR.

Why does it work?

The equilateral triangle GAC makes angle GAC = 60º. Hence angle GBA in triangle GBA must be 30º.

Look at the construction algebraically in Cartesian co-odinates with point C (0, 0), point A (1,0), and so on.

Angle HBA is 30º, so, if we call BH = X, we have for the co-ordinates of H, (Xcos(30º) –1, X/2).
By construction, FH = 3/2.
Since the co-ordinates of point F, (–1/2,0) and point H, (X√3/2 –1, X/2), the distance FH is:

$3/2 = \sqrt{((X\sqrt{3}/2 - 1/2)^2 + (X/2)^2}$

Solving for X gives X = (√3 + √35)/4 = 1.912032648.

If we call the angle JCA, Φ, then the line CJ algebraically is: $y = x \tan(\Phi)$.
Because the line AH passes through both point A and point H, algebraically it is:
$y = mx - m$, where $m = X/(X\sqrt{3} - 4)$.

The intersection of lines CJ and AH at J occurs when $x_J \tan(\Phi) = mx_J - m$
$x_J = -m/(\tan(\Phi) - m)$
$y_J = mx_J - m$

Since CJ = 1
$1 = x_J^2 + y_J^2 = \{m/(\tan(\Phi) + m)\}^2 + \{mx_J - m\}^2$

Solving for $\tan(\Phi)$ gives:
$\tan(\Phi) = 2m/(1 - m^2) = (\sqrt{3}X^2 - 4X)/(X^2 - 4\sqrt{3}X + 8)$

For X = 1.912032648, $\tan(\Phi)$ = 0.827097579; and
$\Phi = 39.594^\circ$

Hence:
AJ = KL = LM = NP = PQ = Φ for a total 5Φ
JK = MN = QR = $2(60^\circ - \Phi) = 40.812^\circ$ for a total of $360^\circ - 6\Phi$
Hence AR = Φ.

The biggest error is thus +0.8º and the smallest is –0.4º.

There's a slightly more accurate method on page 20.

Given that cos 360º/7 is close to 5/8, you might imagine there was a similar simple algebraic expression for cos 360º/9 (cos 40º), but there doesn't seem to be.

If A =360º/9, then cos A is a root of degree-9, trignometric polynomial equation:
$256 \cos^9 A - 576 \cos^7 A + 432 \cos^5 A - 120 \cos^3 A + 9 \cos A - 1 = 0$

which we can reduce, as we did for the degree-7 polynomial, to:

$(\cos A - 1)(16\cos^4 A + 8\cos^3 A - 12\cos^2 A - 4\cos A + 1)^2 = 0$

This leaves us with a quartic equation, and if you think cubic equations are hard to solve, don't try to solve a quartic. Fortunately, we have an "out" in this case because we know that:
$\cos 3A$ ($A=120^\circ$) = –1/2, and $\cos 3A = 4\cos^3 A - 3\cos A$. Hence we need to solve only:

$8\cos^3 A - 6\cos A + 1 = 0$

exact solutions for which are A = 40º, 80º, and 160º. A good approximate non-trignometric solution happens to be $\cos A = \sqrt[3]{0.45} = 3/\sqrt[3]{60}$ (A = 39.976º).

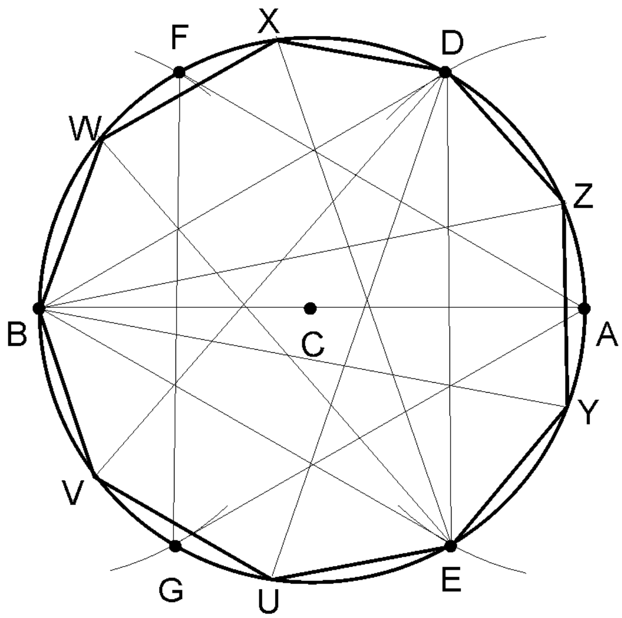
I'm not sure how this helps geometrically, but I suspect, if there is a way, it involves dividing an angle by three which is not possible with simple tools.

The rational numbers that are close to a root begin with:
$\cos A = 3/\sqrt[3]{64} = 3/4$, which is not a good approximation (A = 41.4º).

From there you have to go to
cos A = 10/13 (A = 39.7º), 13/17 (A = 40.12º), 23/30 (A = 39.94º), etc. which is increasingly unhelpful when you are messing around at the dinner table.

NINE (approximate)

Divide a circle into 9 equal parts using only a straight-edge and compass. This method is not as accurate as the previous method, but it's relatively easy to construct.



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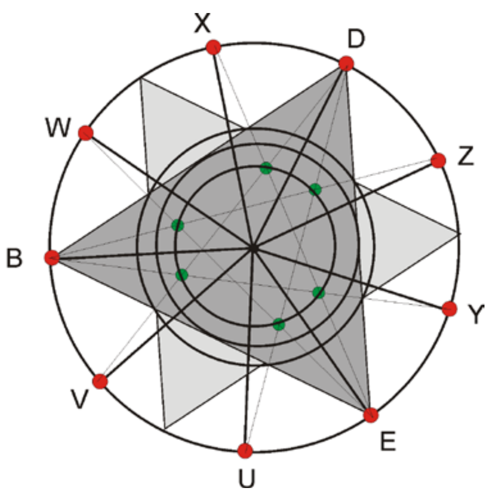
1. Draw the diameter BCA.
2. With compass set to CA, centre A, locate D and E.
3. With compass set again to CA, centre B, locate F and G.
4. Draw the two equilateral triangles AFG and BDE.
5. Starting at B, draw lines from B through the intersection of the two triangles. These are BZ and BY.
6. Repeat step 5 for E, creating EX and EW.
7. Repeat step 5 for D, creating DV and DU.
8. Points Z, D, X, W, B, V, U, E, Y, Z identify the ends of chords dividing the circle by nine.

Why doesn't it work?

There is so much symmetry in this method that one would think that it would work, but although chords YZ, WX, and VU are clearly the same, as are ZD, DX, WB, BV, UE, and EY, these two sets are not exactly equal. You can see this if you compare, for example, triangles ZBY with a diameter BA dividing it equally into two, and AWB where the division into two would not be by a diameter. Too bad.



The petroglyph connection

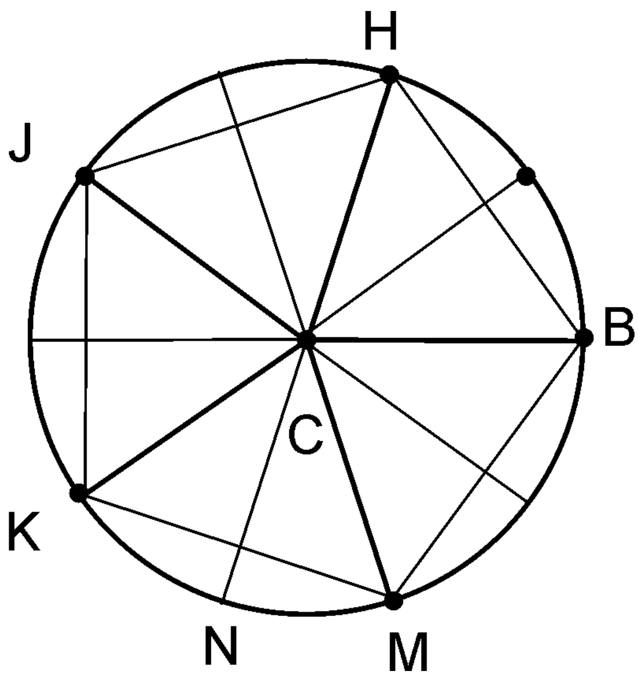
There is a strong possibility that this was the method used by a petroglyph carver on Gabriola Island at site DgRw228. The three inner-circles have been carved as well as seven of the nine "petals". The petroglyph is a calendar dividing the year into nine "months" of 40 days plus a few.

The outer inner-circle marks the intersections of the two equilateral triangles. The middle inner-circle exactly fits inside the two equilateral triangles. And the inner inner-circle marks intersections of the rays BZ, BY, EX, EW, DV, and DU, none of which are located by one triangle alone.

The petroglyph also has a fourth inner-circle, that is smaller than the others and half the size of the largest one.

TEN

Divide a circle into 10 equal parts using only a straight-edge and compass.



1. Divide the circle into five, HJKMB.
2. Extend radial HC to locate N.
3. Repeat step 2 for radials JC, KC, MC, and BC.

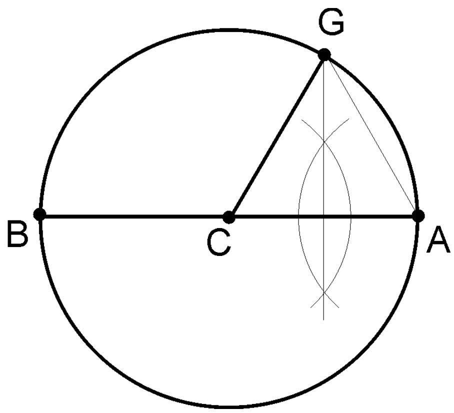
Why does it work?

Each 72º angle of the pentagon is split into two 36º angles, making ten in all.

ELEVEN

Divide a circle into 11 equal parts using only a straight-edge and compass. This is technically not possible, and even good approximate methods are complicated. I'll do this one in steps.

STEP 1



1. Draw the diameter of the circle B through C to A.
2. With compass set to more than half CA, draw arcs centre C and centre A and put a line through their intersections to locate G.

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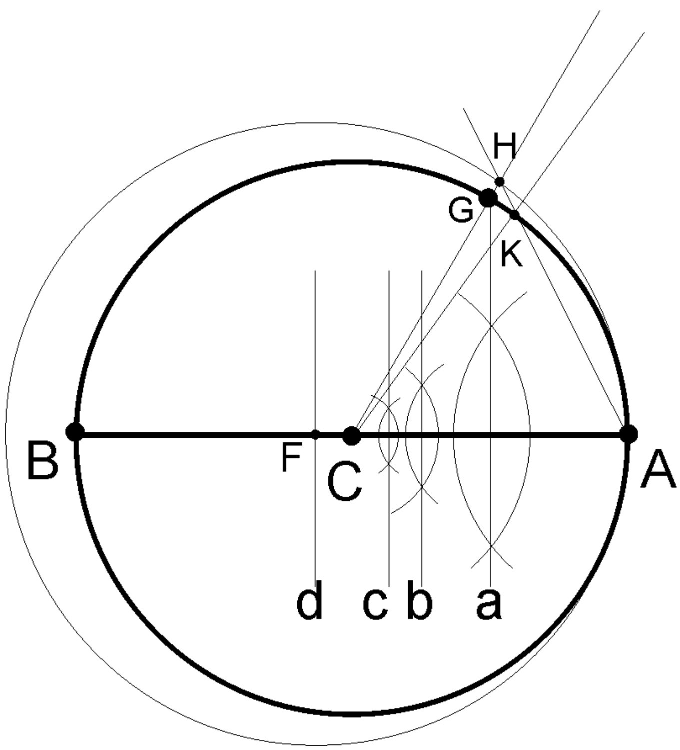
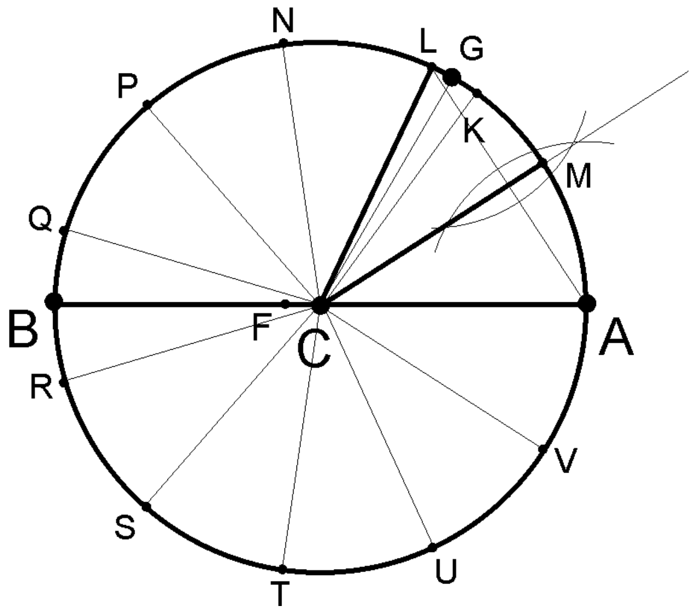


STEP 2

The letters a, b, c, d represent points on the diameter BA.

1. With compass set to more than half Ca on CA, draw arcs centre C and centre "a" and put a line through their intersections to find "b" and to divide CA by four.
2. With compass set to more than half Cb on CA, draw arcs centre C and centre "b" and put a line through their intersections to find "c" and to divide CA by eight.
3. Locate F on the left of C on CB at a distance Cc on CA from C.
4. Draw a circle centre F, radius FA.
5. Locate H on this second circle by extending CG.
6. Draw HA and locate K on the first circle.

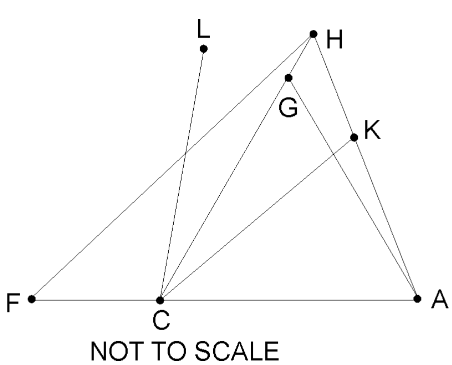


STEP 3

1. With compass centre G, set to GK, locate L on the other side of G.
2. With compass set to more than half LA, draw arcs centre A and centre L and put a line through their intersections to locate M.
3. Walk LM round the circle to N, P, Q, R, S, T, U, and V.
4. These points divide the circle almost exactly, but not exactly, by 11.

Why does it work?

I don't know to be honest but I suspect it divides the circle by twelve and then divides the 12th slice by eleven for re-distribution. An 11th of a 12th is so small an angle θ that there is a trivial difference between the arc length Rθ, and the corresponding straight-line chord length, 2Rsin(θ/2). It's cheating really.



All I can do though is calculate the value of the angle MCL and show that it is close to being 360º/11.

Call the radius CA, 1 unit of distance.

The equilateral triangle CGA makes angle GCA = 60º. Hence angle GCF = angle HCF = 120º.

In triangle HFC, sin(CHF) = (FC/HF)sin(HCF) = (FC/HF) √3/2 = (1/8)/(9/8) √3/2 = 1/6√3
Hence angle CHF = 5.52183º and angle HFA = 54.47817º

In the isosceles triangle HFA, HA/2 = HF sin(HFA/2) =9/8 sin(27.23908º). Hence HA = 1.029835.

In triangle CHA, sin(CHA) = (CA/HA)sin(HCA) = (1/1.029835) √3/2. Hence angle CHA = 57.23910º.

In triangle AGH, angle AGH = 120º, therefore angle HAG = 60º – CHA = 2.7609º.

Since angle GAC = 60º, angle KAC = 60º + 2.7609º = 62.7609º

Since triangle KCA is isosceles, angle KCA = 180º – 2(KAC) = 54.47817º.

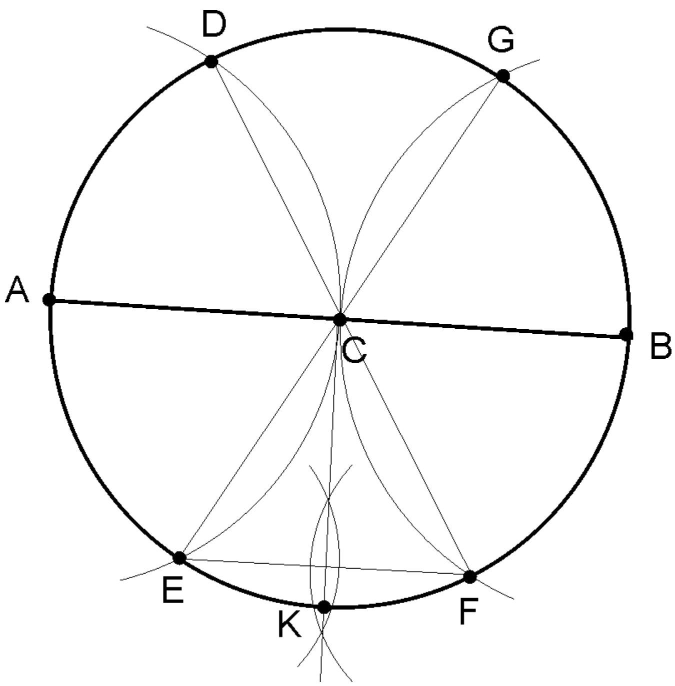
Hence angle LCA = 60º + (60º – 54.47817º) = 65.52183º and angle MCA = LCA/2 = 32.7609º.

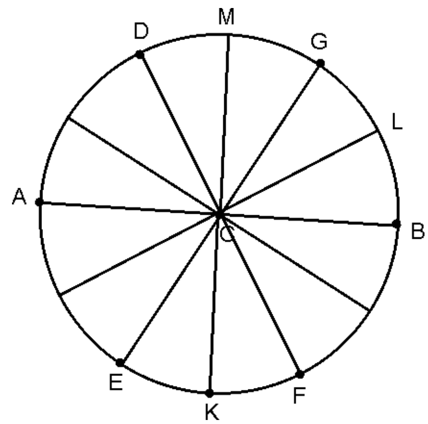
Eleven times angle MCA is thus 360.37º and the biggest error is –0.4º and the smallest is –0.03º.

There's probably a more elegant method of calculating this, but I'm done.

TWELVE

Divide a circle into 12 equal parts using only a straight-edge and compass.





1. Draw diameter AB through the centre C.
2. With compass centre A set with radius AC, find D and E.
3. With compass centre B set with the same setting, find G and F.
4. Bisect EF with arcs centre E and centre F, to locate K. Extend KC to meet the circle at M.
5. Repeat step 4 for chords FB and BG.

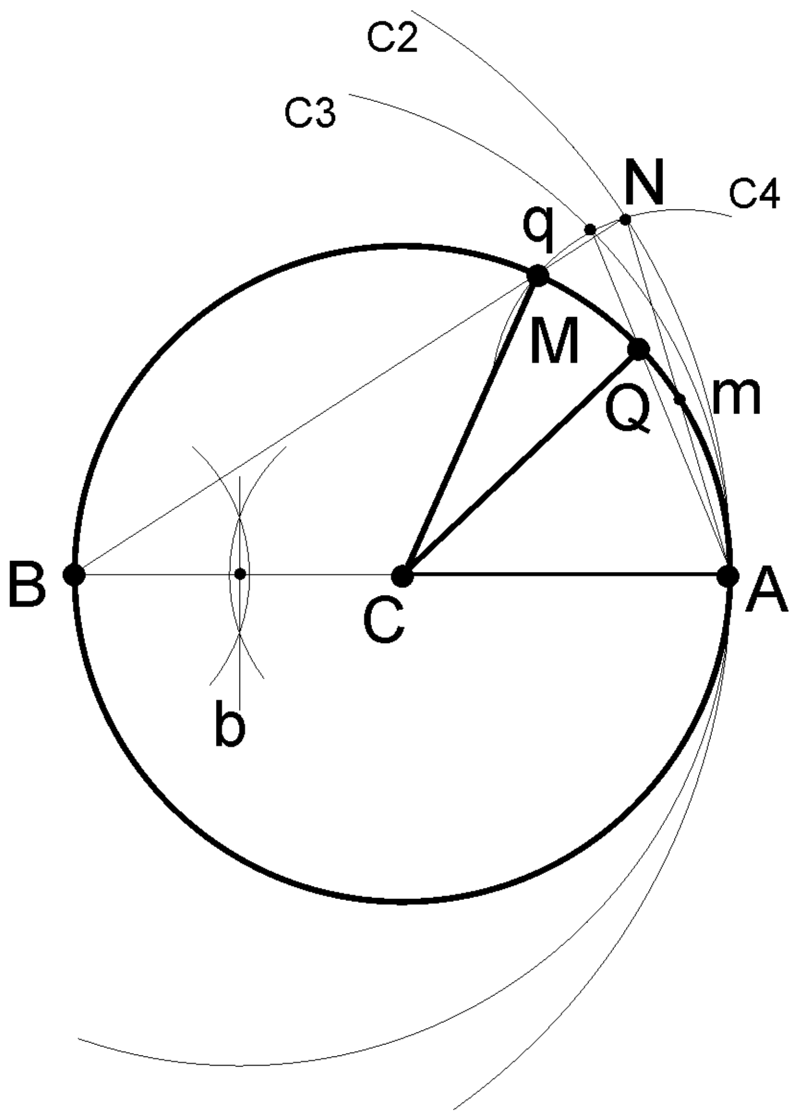
Why does it work?

ECF is an equilateral triangle. Dividing its internal angle of 60º by two creates two 30º segments. Doing this six times divides the circle equally by twelve.

DIVISION OF A LEFT-OVER SLICE BY THREE

Dividing a left-over slice by 2, 4, or 8 is easy, but to divide it by 3, 6, or 12, you have to be able to divide any arbitrary angle by three using only a straight-edge and compass. This is the John Harris method of approximately doing that.

I'm not going to attempt 5 (and hence 10), or 7 and 11.



1. Start with the slice to be divided MCA. We want angle QCA = 2/3 angle MCA. Once that is done, dividing chord QA by 2 and drawing from C through the dividing point (not shown) will then produce three equal slices.
2. With compass radius CA, centre C, draw the original circle, and draw diameter BCA.
3. With compass set to more than half BC, draw arcs centre C and centre B and put a line through their intersections to locate b on BC.
4. With compass set to BA, centre B, draw arc C2.
5. With compass set to bA, centre b, draw arc C3.
6. Draw BM and extend it to circle C2 at N.
7. Join NA and locate m on the original circle.
8. With compass set to mN. centre m, draw arc C4.
9. Locate q at the intersection of arc C4 and C3.
10. Draw Aq and locate Q on the original circle. Join CQ.

Why does it work?

Call the radius CA, 1 unit of distance. Call angle MCA angle Φ. In the drawing Φ = 66º, and I'll indicate numerical values for this particular value of Φ without loss of generality.

In triangle MCB, angle MCB = 180º – Φ {114º}

Triangle MCB is isosceles, so angle MBC =1/2(180º – (180º – Φ)) = Φ/2 {57º}

Angle NBA = angle MBC = Φ/2

Triangle NBA is isosceles, so NA = 2(2 sin(NBA/2)) = 4 sin(Φ/4) {1.136061…}

Angle NAB = NAC = mAC = 1/2(180º – Φ/2) = 90º – Φ/4 {73.5º}

Triangle mCA is isosceles, so mA= 2 CA cos(mAC) = 2 sin(Φ/4)

Hence mN = 4 sin(Φ/4) – 2sin(Φ/4) = mA {0.568031…}

I have only done the next steps numerically.
The (x,y) coordinates of m relative to C (0,0) are:
(CA – mA cos(mAC), mA sin(mAC)) = (cos(Φ/2), sin(Φ/2))
Hence the equation of C4 is: $(y - \sin(\Phi/2))^2 + (x - \cos(\Phi/2))^2 = 4\sin^2(\Phi/4)$

The (x,y) coordinates of b relative to C (0,0) are (-1/2, 0)
Hence the equation of C3 is: $y^2 + (x + 0.5)^2 = 1.5^2$

At q, the intersection of C4 and C3, and for Φ = 66º, the co-ordinates are:
X_q = 0.573940774, Y_q = 1.047209586, $m_{qA} = Y_q/(X_q - 1)$ = –2.4579

Hence the line qA is: $y = m_{qA}(x - 1)$; and
the line CQ is: y = tan(QCA) x.

These intersect at : X_Q = 0.573941598 and Y_Q = 1.047210001 and hence QCA = 44.278º

Not bad, but perfection would have been QCA = 44º.

Dividing by nine again

For Φ = 60º, QCA = 40.208º

so you could use this method for dividing a pie (or pizza) by nine. However, if you walk 40.208º around the circle 8 times, you are left with a slice of only 360 – 8 x 40.208º = 38.337º. Definitely not OK; the host is short-changed by 1.66º .

What you need for this task is an additional angle that is slightly less than 40º, such as twice angle QCM, which is = 39.584º. Six slices of 40.208º and three slices of 39.584º makes up a full circle with the greatest error –0.416º and the smallest +0.208º. A slight improvement on the method given on page 12 for dividing by nine. ◊